# FORMALITY, ALEXANDER INVARIANTS, AND A QUESTION OF SERRE

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Abstract. We elucidate the key role played by formality in the theory of characteristic and resonance varieties. We show that the $I$-adic completion of the Alexander invariant of a 1-formal group $G$ is determined solely by the cup-product map in low degrees. It follows that the germs at the origin of the characteristic and resonance varieties of $G$ are analytically isomorphic; in particular, the tangent cone to $V_k(G)$ at 1 equals $R_k(G)$. This provides new obstructions to 1-formality. A detailed analysis of the irreducible components of the tangent cone at 1 to the first characteristic variety yields powerful obstructions to realizing a finitely presented group as the fundamental group of a smooth, complex quasi-projective algebraic variety. This sheds new light on a classical problem of J.-P. Serre. Applications to arrangements, configuration spaces, coproducts of groups, and Artin groups are given.

Contents



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## 1. Introduction

A recurring theme in algebraic topology and geometry is the extent to which the cohomology of a space reflects the underlying topological or geometric properties of that space. In this paper, we focus on the degree-one cohomology jumping loci of the fundamental group $G$ of a connected, finite-type CW-complex $M$: the characteristic varieties $\mathcal{V}_k(G)$, and the resonance varieties $\mathcal{R}_k(G)$. Our goal is to relate these two sets of varieties, and to better understand their structural properties, under certain naturally defined conditions on $M$. In turn, this analysis yields new and powerful obstructions for a finitely presented group $G$ to be 1-formal, or realizable as the fundamental group of a smooth, complex quasi-projective variety.

1.1. **Cohomology jumping loci.** Let $\mathbb{T}_G = \operatorname{Hom}(G, \mathbb{C}^*)$ be the character variety of $G = \pi_1(M)$. The *characteristic varieties* of $M$ are the jumping loci for the cohomology of $M$, with coefficients in rank 1 local systems:

$$\mathcal{V}^i_k(M) = \{\rho \in \mathbb{T}_G \mid \dim H^i(M, {}_\rho\mathbb{C}) \geq k\}. \tag{1.1}$$

These varieties emerged from the work of S. Novikov [56] on Morse theory for closed 1-forms on manifolds. Foundational results on the structure of the cohomology support loci for local systems on quasi-projective algebraic varieties were obtained by Beauville [3, 4], Green and Lazarsfeld [31], Simpson [66, 67], Arapura [2], and Campana [8].

Let $H^\bullet(M, \mathbb{C})$ be the cohomology algebra of $M$. Right-multiplication by an element $z \in H^1(M, \mathbb{C})$ yields a cochain complex $(H^\bullet(M, \mathbb{C}), \mu_z)$. The *resonance varieties* of $M$ are the jumping loci for the homology of this complex:

$$\mathcal{R}^i_k(M) = \{z \in H^1(M, \mathbb{C}) \mid \dim H^i(H^\bullet(M, \mathbb{C}), \mu_z) \geq k\}. \tag{1.2}$$

These homogeneous algebraic subvarieties of $H^1(M, \mathbb{C}) = \operatorname{Hom}(G, \mathbb{C})$ were first defined by Falk [28] in the case when $M$ is the complement of a complex hyperplane arrangement; in this setting, a purely combinatorial description of $\mathcal{R}^1_k(M)$ was given by Falk [28], Libgober–Yuzvinsky [45], and Falk–Yuzvinsky [29].

We consider here only the cohomology jumping loci in degree $i = 1$. These loci depend exclusively on $G = \pi_1(M)$, so we write $\mathcal{V}_k(G) = \mathcal{V}^1_k(M)$ and $\mathcal{R}_k(G) = \mathcal{R}^1_k(M)$. The higher degree jumping loci will be treated in a forthcoming paper.

1.2. **The tangent cone theorem.** The key topological property that allows us to relate the characteristic and resonance varieties of a space $M$ is *formality*, in the sense of D. Sullivan [68]. Since we deal solely with the fundamental group $G = \pi_1(M)$, we only need $G$ to be 1-*formal*. This property requires that $E_G$, the Malcev Lie algebra of $G$ (over $\mathbb{C}$), be isomorphic, as a filtered Lie algebra, to the holonomy Lie algebra $\mathfrak{H}(G) = \mathbb{L}/\langle \operatorname{im} \partial_G \rangle$, completed with respect to bracket length filtration. Here $\mathbb{L}$ denotes the free Lie algebra on $H_1(G, \mathbb{C})$, while $\partial_G$ denotes the dual of the cup-product map $\cup_G\colon \bigwedge^2 H^1(G, \mathbb{C}) \to H^2(G, \mathbb{C})$.

A group $G$ is 1-formal if and only if $E_G$ can be presented with quadratic relations only; see Section 2 for details. Many interesting groups fall in this class: knot groups ([68]), certain pure braid groups of Riemann surfaces (Bezrukavnikov [6], Hain [33]), fundamental groups of compact Kähler manifolds (Deligne, Griffiths, Morgan, and Sullivan [18]), fundamental groups of complements of hypersurfaces in $\mathbb{CP}^n$ (Kohno [39]), and finite-type Artin groups (Kapovich–Millson [38]) are all 1-formal.

**Theorem A.** *Let $G$ be a 1-formal group. For each $0 \le k \le b_1(G)$, the exponential map* $\exp\colon \operatorname{Hom}(G,\mathbb{C}) \to \operatorname{Hom}(G,\mathbb{C}^*)$ *restricts to an isomorphism of analytic germs,*

$$\exp\colon (\mathcal{R}_k(G),0) \xrightarrow{\simeq} (\mathcal{V}_k(G),1).$$

*In particular, the tangent cone at 1 to $\mathcal{V}_k(G)$ equals $\mathcal{R}_k(G)$:*

$$TC_1(\mathcal{V}_k(G)) = \mathcal{R}_k(G)\,.$$

Essential ingredients in the proof are two modules associated to a finitely presented group $G$: the *Alexander invariant*, $B_G = G'/G''$, viewed as a module over the group ring $\mathbb{Z}G_{\mathrm{ab}}$, and its "infinitesimal" version, $B_{\mathfrak{H}(G)} = \operatorname{coker}(\delta_3+\partial_G)$, viewed as a module over the polynomial ring $\mathbb{C}[X] = \operatorname{Sym}(G_{\mathrm{ab}} \otimes \mathbb{C})$, where $\delta_j$ denotes the $j$-th Koszul differential. The varieties defined by the Fitting ideals of these two modules coincide, away from the origin, with $\mathcal{V}_k(G)$ and $\mathcal{R}_k(G)$, respectively.

Under our 1-formality assumption on $G$, we deduce Theorem A from the fact that the exponential map induces a filtered isomorphism between the $I$-adic completion of $B_G \otimes \mathbb{C}$ and the $(X)$-adic completion of $B_{\mathfrak{H}(G)}$. We establish this isomorphism using Malcev Lie algebra tools. Related techniques have been used previously in low-dimensional topology [33, 58], algebraic geometry [55, 1], and group theory [48, 60].

Theorem A sharpens and extends several results from [27, 64, 14], which only apply to the case when $G$ is the fundamental group of the complement of a complex hyperplane arrangement. The point is that only a *topological* property (1-formality) is needed for the conclusion to hold. Further information in the case of hypersurface arrangements can be found in [43, 21].

For an arbitrary finitely presented group $G$, the tangent cone to $\mathcal{V}_k(G)$ at the origin is contained in $\mathcal{R}_k(G)$, see Libgober [44]. Yet the inclusion may well be strict. In fact, as noted in Example 5.11 and Remark 9.3, the tangent cone formula from Theorem A fails even for fundamental groups of smooth, quasi-projective varieties.

Theorem A provides a new, and quite powerful obstruction to 1-formality of groups—and thus, to formality of spaces. As illustrated in Example 8.2, this obstruction complements, and in some cases strengthens, classical obstructions to (1-)formality, due to Sullivan, such as the existence of an isomorphism $\operatorname{gr}(G)\otimes\mathbb{C} \cong \mathfrak{H}(G)$.

1.3. **Serre's question.** As is well-known, every finitely presented group $G$ is the fundamental group of a smooth, compact, connected 4-dimensional manifold $M$. Requiring a complex structure on $M$ is no more restrictive, as long as one is willing to go up in dimension; see Taubes [69]. Requiring that $M$ be a compact Kähler

manifold, though, puts extremely strong restrictions on what $G = \pi_1(M)$ can be. We refer to [1] for a comprehensive survey of Kähler groups.

J.-P. Serre asked the following question: which finitely presented groups can be realized as fundamental groups of smooth, connected, quasi-projective, complex algebraic varieties? Following Catanese [10], we shall call a group $G$ arising in this fashion a *quasi-projective* group.

In this context, one may also consider the larger class of quasi-compact Kähler manifolds, of the form $M = \overline{M} \setminus D$, where $\overline{M}$ is compact Kähler and $D$ is a normal crossing divisor. If $G = \pi_1(M)$ with $M$ as above, we say $G$ is a *quasi-Kähler* group.

The first obstruction to quasi-projectivity was given by J. Morgan: If $G$ is a quasi-projective group, then $E_G = \widehat{\mathbb{L}}/J$, where $\mathbb{L}$ is a free Lie algebra with generators in degrees 1 and 2, and $J$ is a closed Lie ideal, generated in degrees 2, 3 and 4; see [55, Corollary 10.3]. By refining Morgan's techniques, Kapovich and Millson obtained analogous quasi-homogeneity restrictions, on certain singularities of representation varieties of $G$ into reductive algebraic Lie groups; see [38, Theorem 1.13]. Another obstruction is due to Arapura: If $G$ is quasi-Kähler, then the characteristic variety $\mathcal{V}_1(G)$ is a union of (possibly translated) subtori of $\mathbb{T}_G$; see [2, p. 564].

If the group $G$ is 1-formal, then $E_G = \widehat{\mathbb{L}}/J$, with $\mathbb{L}$ generated in degree 1 and $J$ generated in degree 2; thus, $G$ verifies Morgan's test. It is therefore natural to explore the relationship between 1-formality and quasi-projectivity. (In contrast with the Kähler case, it is known from [55, 38] that these two properties are independent.) Another motivation for our investigation comes from the study of fundamental groups of complements of plane algebraic curves. This class of 1-formal, quasi-projective groups contains, among others, the celebrated Stallings group; see [62].

1.4. **Position and resonance obstructions.** Our main contribution to Serre's problem is Theorem B below, which provides a new type of restriction on fundamental groups of smooth, quasi-projective complex algebraic varieties. In the presence of 1-formality, this restriction is expressed entirely in terms of a classical invariant, namely the cup-product map $\cup_G\colon \bigwedge^2 H^1(G,\mathbb{C}) \to H^2(G,\mathbb{C})$.

**Theorem B.** *Let $M$ be a connected, quasi-compact Kähler manifold. Set $G = \pi_1(M)$ and let $\{\mathcal{V}^\alpha\}$ be the irreducible components of $\mathcal{V}_1(G)$ containing 1. Denote by $\mathcal{T}^\alpha$ the tangent space at 1 of $\mathcal{V}^\alpha$. Then the following hold.*

1. *Every positive-dimensional tangent space $\mathcal{T}^\alpha$ is a p-isotropic linear subspace of $H^1(G,\mathbb{C})$, of dimension at least $2p+2$, for some $p = p(\alpha) \in \{0,1\}$.*
2. *If $\alpha \neq \beta$, then $\mathcal{T}^\alpha \bigcap \mathcal{T}^\beta = \{0\}$.*

*Assume further that $G$ is 1-formal. Let $\{\mathcal{R}^\alpha\}$ be the irreducible components of $\mathcal{R}_1(G)$. Then the following hold.*

3. *The collection $\{\mathcal{T}^\alpha\}$ coincides with the collection $\{\mathcal{R}^\alpha\}$.*
4. *For $1 \leq k \leq b_1(G)$, we have $\mathcal{R}_k(G) = \{0\} \cup \bigcup_\alpha \mathcal{R}^\alpha$, where the union is over all components $\mathcal{R}^\alpha$ such that $\dim \mathcal{R}^\alpha > k + p(\alpha)$.*

(5) *The group $G$ has a free quotient of rank at least 2 if and only if $\mathcal{R}_1(G)$ strictly contains $\{0\}$.*

Here, we say that a non-zero subspace $V \subseteq H^1(G,\mathbb{C})$ is 0- (respectively, 1-) *isotropic* if the restriction of the cup-product map, $\cup_G\colon \bigwedge^2 V \to \cup_G(\bigwedge^2 V)$, is equivalent to $\cup_C\colon \bigwedge^2 H^1(C,\mathbb{C}) \to H^2(C,\mathbb{C})$, where $C$ is a non-compact (respectively, compact) smooth, connected complex curve. See 5.6 for a more precise definition.

In this paper, we consider only components of $\mathcal{V}_1(G)$ containing 1. For a detailed analysis of translated components, we refer to [20] and [24].

The proof of Theorem B is given in Section 6, and relies on results of Arapura [2] on quasi-Kähler groups. Part (1) is an easy consequence of [2, Proposition V.1.7]. Part (2) is a new viewpoint, developed in [24] to obtain a completely new type of obstruction. Part (3) follows from our tangent cone formula.

For an arrangement complement $M$, Parts (2) and (4) of the above theorem were proved by Libgober and Yuzvinsky in [45], using completely different methods.

The "position" obstructions (1) and (2) in Theorem B can be viewed as a much strengthened form of Arapura's theorem: they give information on how the components of $\mathcal{V}_1(G)$ passing through the origin intersect at that point, and how their tangent spaces at 1 are situated with respect to the cup-product map $\cup_G$.

We may also view conditions (1)–(5) as a set of "resonance" obstructions for a 1-formal group to be quasi-Kähler, or for a quasi-Kähler group to be 1-formal. Since the class of homotopy types of compact Kähler manifolds is strictly larger than the class of homotopy types of smooth projective varieties (see Voisin [72]), one may wonder whether the class of quasi-Kähler groups is also strictly larger than the class of quasi-projective groups.

1.5. **Applications.** In the last four sections, we illustrate the efficiency of our obstructions to 1-formality and quasi-Kählerianity with several classes of examples.

We start in Section 7 with wedges and products of spaces. Our analysis of resonance varieties of wedges, together with Theorem B, shows that 1-formality and quasi-Kählerianity behave quite differently with respect to the coproduct operation for groups. Indeed, if $G_1$ and $G_2$ are 1-formal, then $G_1 * G_2$ is also 1-formal; but, if in addition both factors are non-trivial, presented by commutator relators only, and one of them is non-free, then $G_1 * G_2$ is not quasi-Kähler. As a consequence of the position obstruction from Theorem B(1), we also show that the quasi-Kählerianity of $G_1 * G_2$, where the groups $G_i$ are assumed only finitely presented with infinite abelianization, implies the vanishing of $\cup_{G_1}$ and $\cup_{G_2}$.

When it comes to resonance varieties, real subspace arrangements offer a stark contrast to complex hyperplane arrangements, cf. [51, 52]. If $M$ is the complement of an arrangement of transverse planes through the origin of $\mathbb{R}^4$, then $G = \pi_1(M)$ passes Sullivan's gr-test. Yet, as we note in Section 8, the group $G$ may fail the

tangent cone formula from Theorem A, and thus be non-1-formal; or, $G$ may be 1-formal, but fail tests (1), (2), (4) from Theorem B, and thus be non-quasi-Kähler.

In Section 9, we apply our techniques to the configuration spaces $M_{g,n}$ of $n$ ordered points on a closed Riemann surface of genus $g$. Clearly, the surface pure braid groups $P_{g,n} = \pi_1(M_{g,n})$ are quasi-projective. On the other hand, if $n \geq 3$, the variety $\mathcal{R}_1(P_{1,n})$ is irreducible and non-linear. Theorem B(3) shows that $P_{1,n}$ is not 1-formal, thereby verifying a statement of Bezrukavnikov [6].

We conclude in Section 10 with a study of Artin groups associated to finite, labeled graphs, from the perspective of their cohomology jumping loci. As shown in [38], all Artin groups are 1-formal; thus, they satisfy Morgan's homogeneity test. Moreover, as we show in Proposition 10.5 (building on work from [61]), the first characteristic varieties of right-angled Artin groups are unions of coordinate subtori; thus, all such groups pass Arapura's $\mathcal{V}_1$-test.

In [38, Theorem 1.1], Kapovich and Millson establish, by a fairly involved argument, the existence of infinitely many Artin groups that are not realizable by smooth, quasi-projective varieties. Using the isotropicity obstruction from Theorem B, we show that a right-angled Artin group $G_\Gamma$ is quasi-Kähler if and only if $\Gamma$ is a complete, multi-partite graph, in which case $G_\Gamma$ is actually quasi-projective. This result provides a complete—and quite satisfying—answer to Serre's problem within this class of groups. In the process, we take a first step towards solving the problem for all Artin groups, by answering it at the level of associated Malcev Lie algebras. We also determine precisely which right-angled Artin groups are Kähler.

The approach to Serre's problem taken in this paper—based on the obstructions from Theorem B—has led to complete answers for several other classes of groups:

- In [22], we classify the quasi-Kähler groups within the class of groups introduced by Bestvina and Brady in [5].
- In [24], we determine precisely which fundamental groups of boundary manifolds of line arrangements in $\mathbb{CP}^2$ are quasi-projective groups.
- In [25], we decide which 3-manifold groups are Kähler groups, thus answering a question raised by S. Donaldson and W. Goldman in 1989.
- In [59], we show that the fundamental groups of a certain natural class of graph links in $\mathbb{Z}$-homology spheres are quasi-projective if and only if the corresponding links are Seifert links.

The obstructions from Theorem B are complemented by new methods of constructing interesting (quasi-)projective groups. These methods, developed in [22] and [23], lead to a negative answer to the following question posed by J. Kollár in [40]: Is every projective group commensurable (up to finite kernels) with a group admitting a quasi-projective variety as classifying space?

## 2. Holonomy Lie algebra, Malcev completion, and 1-formality

Given a (finitely presented) group $G$, there are several Lie algebras attached to it: the associated graded Lie algebra $\mathrm{gr}^*(G)$, the holonomy Lie algebra $\mathfrak{H}(G)$, and the Malcev Lie algebra $E_G$. In this section, we review the constructions of these Lie algebras, and the related notion of 1-formality, which will be crucial in what follows.

2.1. **Holonomy Lie algebras.** We start by recalling the definition of the holonomy Lie algebra, due to K.-T. Chen [12].

Let $M$ be a connected CW-complex with finite 2-skeleton. Let $\Bbbk$ be a field of characteristic 0. Denote by $\mathbb{L}^*(H_1M)$ the free Lie algebra on $H_1M = H_1(M,\Bbbk)$, graded by bracket length, and use the Lie bracket to identify $H_1M \wedge H_1M$ with $\mathbb{L}^2(H_1M)$. Set

$$\mathfrak{H}(M) := \mathbb{L}^*(H_1M)/\langle \mathrm{im}(\partial_M \colon H_2M \to \mathbb{L}^2(H_1M))\rangle, \tag{2.1}$$

where $\partial_M$ is the dual of the cup-product map $\cup_M \colon H^1(M,\Bbbk)\wedge H^1(M,\Bbbk) \to H^2(M,\Bbbk)$ and $\langle(\cdot)\rangle$ denotes the Lie ideal spanned by $(\cdot)$. Note that $\mathfrak{H}(M)$ is a quadratic Lie algebra, in that it has a presentation with generators in degree 1 and relations in degree 2 only. We call $\mathfrak{H}(M)$ the *holonomy Lie algebra* of $M$ (over the field $\Bbbk$).

Now let $G$ be a group admitting a finite presentation. Choose an Eilenberg-MacLane space $K(G,1)$ with finite 2-skeleton, and define

$$\mathfrak{H}(G) := \mathfrak{H}(K(G,1)). \tag{2.2}$$

If $M$ is a CW-complex as above, with $G = \pi_1(M)$, and if $f\colon M \to K(G,1)$ is a classifying map, then $f$ induces an isomorphism on $H_1$ and an epimorphism on $H_2$. This implies that

$$\mathfrak{H}(G) = \mathfrak{H}(M). \tag{2.3}$$

2.2. **Malcev Lie algebras.** Next, we recall some useful facts about the functorial Malcev completion of a group, following Quillen [63, Appendix A].

A *Malcev Lie algebra* is a $\Bbbk$-Lie algebra $E$, endowed with a decreasing, complete filtration

$$E = F_1E \supset \cdots \supset F_qE \supset F_{q+1}E \supset \cdots,$$

by $\Bbbk$-vector subspaces satisfying $[F_rE, F_sE] \subset F_{r+s}E$ for all $r,s \geq 1$, and with the property that the associated graded Lie algebra, $\mathrm{gr}^*_F(E) = \bigoplus_{q\geq 1} F_qE/F_{q+1}E$, is generated by $\mathrm{gr}^1_F(E)$. By completeness of the filtration, the Campbell-Hausdorff formula

$$e\cdot f = e + f + \tfrac{1}{2}[e,f] + \tfrac{1}{12}([e,[e,f]] + [f,[f,e]]) + \cdots \tag{2.4}$$

endows the underlying set of $E$ with a group structure, to be denoted by $\mathsf{exp}(E)$.

For a group $G$, denote by $\widehat{\mathfrak{H}}(G)$ the completion of the holonomy Lie algebra with respect to the degree filtration. It is readily checked that $\widehat{\mathfrak{H}}(G)$, together with the

canonical filtration of the completion, is a Malcev Lie algebra with $\mathrm{gr}^*_F(\widehat{\mathfrak{H}}(G)) = \mathfrak{H}^*(G)$.

In [63], Quillen associates to $G$, in a functorial way, a Malcev Lie algebra $E_G$ and a group homomorphism $\kappa_G\colon G \to \mathsf{exp}(E_G)$. The key property of the Malcev completion $(E_G, \kappa_G)$ is that $\kappa_G$ induces an isomorphism of graded $\Bbbk$-Lie algebras

$$\mathrm{gr}^*(\kappa_G)\colon\ \mathrm{gr}^*(G) \otimes \Bbbk \xrightarrow{\simeq} \mathrm{gr}^*_F(E_G). \tag{2.5}$$

Here, $\mathrm{gr}^*(G) = \bigoplus_{q\geq 1} \Gamma_q G/\Gamma_{q+1}G$ is the graded Lie algebra associated to the lower central series, $G = \Gamma_1 G \supset \cdots \supset \Gamma_q G \supset \Gamma_{q+1} G \supset \cdots$, defined inductively by setting $\Gamma_{q+1}G = (G, \Gamma_q G)$, where $(A, B)$ denotes the subgroup generated by all commutators $(g, h) = ghg^{-1}h^{-1}$ with $g \in A$ and $h \in B$, and with the Lie bracket on $\mathrm{gr}^*(G)$ induced by the commutator map $(\ ,\ )\colon G \times G \to G$.

2.3. **Formal spaces and 1-formal groups.** The important notion of formality of a space was introduced by D. Sullivan in [68]. Let $M$ be a connected CW-complex with finitely many cells in each dimension. Roughly speaking, $M$ is formal if the rational cohomology algebra of $M$ determines the homotopy type of $M$ modulo torsion. Many interesting spaces are formal: compact Kähler manifolds [18], homogeneous spaces of compact connected Lie groups with equal ranks [68]; products and wedges of formal spaces are again formal.

**Definition 2.4.** A finitely presented group $G$ is 1-*formal* if its Malcev Lie algebra, $E_G$, is isomorphic to the completion of its holonomy Lie algebra, $\widehat{\mathfrak{H}}(G)$, as filtered Lie algebras.

A fundamental result of Sullivan [68] states that $\pi_1(M)$ is 1-formal whenever $M$ is formal. The converse is not true in general, though it holds under certain additional assumptions, see [61]. Here are some motivating examples.

**Example 2.5.** If $M$ is obtained from a smooth, complex projective variety $\overline{M}$ by deleting a subvariety $D \subset \overline{M}$ with $\operatorname{codim} D \geq 2$, then $\pi_1(M) = \pi_1(\overline{M})$. Hence, $\pi_1(M)$ is 1-formal, by [18].

**Example 2.6.** Let $W_*(H^m(M, \mathbb{C}))$ be the Deligne weight filtration [17] on the cohomology with complex coefficients of a connected smooth quasi-projective variety $M$. It follows from a basic result of J. Morgan [55, Corollary 10.3] that $\pi_1(M)$ is 1-formal if $W_1(H^1(M, \mathbb{C})) = 0$. By [17, Corollary 3.2.17], this vanishing property holds whenever $M$ admits a non-singular compactification with trivial first Betti number. As noted in [39], these two facts together establish the 1-formality of fundamental groups of complements of projective hypersurfaces.

**Example 2.7.** If $b_1(G) = 0$, it follows from [63] that $E_G \cong \widehat{\mathfrak{H}}(G) = 0$, therefore $G$ is 1-formal. If $G$ is finite, Serre showed in [65] that $G$ is the fundamental group of a smooth complex projective variety.

2.8. **On the Malcev completion of a 1-formal group.** The following test for 1-formality is well-known, and will be useful in the sequel. For the benefit of the reader, we include a proof.

**Lemma 2.9.** *A finitely presented group $G$ is 1-formal if and only if the Malcev Lie algebra $E_G$ is isomorphic, as a filtered Lie algebra, to the completion with respect to degree of a quadratic Lie algebra.*

*Proof.* Write $E_G = \widehat{C}$, with $C^*$ a graded Lie algebra presented as $C^* = \mathbb{L}^*(X)/\langle Y\rangle$, with $Y \subset \mathbb{L}^2(X)$ a $\Bbbk$-vector subspace. It is enough to prove that $C \cong \mathfrak{H}(G)$, as graded Lie algebras.

From [68], we know that that the image of $\partial_G = \partial_{K(G,1)}$ equals the kernel of the Lie bracket map $\bigwedge^2 \mathrm{gr}^1(G)\otimes\Bbbk \to \mathrm{gr}^2(G)\otimes\Bbbk$. From (2.5), we get a graded Lie algebra isomorphism $\mathrm{gr}^*(G)\otimes\Bbbk \cong C^*$. Via this isomorphism, $H_1(G,\Bbbk) = \mathrm{gr}^1(G)\otimes\Bbbk$ is identified to $X$ and $\mathrm{im}(\partial_G)$ to $Y$. Thus, $C^* \cong \mathfrak{H}^*(G)$. □

It is convenient to normalize the Malcev completion of a 1-formal group in the following way. Recall that, for a group $G$ with holonomy Lie algebra $\mathfrak{H}(G)$, we have $\mathrm{gr}^*_F(\widehat{\mathfrak{H}}(G)) = \mathfrak{H}^*(G)$.

**Lemma 2.10.** *If $G$ is a 1-formal group, then there is a Malcev completion homomorphism*

$$\kappa\colon G \to \mathsf{exp}(\widehat{\mathfrak{H}}(G)),$$

*inducing the identity on* $\mathrm{gr}^1(G)\otimes\Bbbk = \mathrm{gr}^1_F(\widehat{\mathfrak{H}}(G)) = \mathfrak{H}^1(G)$.

*Proof.* Set $\mathfrak{H} = \mathfrak{H}(G)$, $X = G_{\mathrm{ab}}\otimes\Bbbk$ and $Y = \mathrm{im}(\partial_G)$. Let $\kappa_G\colon G \to \mathsf{exp}(\widehat{\mathfrak{H}})$ be a Malcev completion, and set $\phi = \mathrm{gr}^1(\kappa_G)\colon X \to X$. It follows from (2.5) that $\phi$ is an isomorphism. Moreover, $\mathbb{L}^2(\phi)\colon \bigwedge^2 X \to \bigwedge^2 X$ is also an isomorphism, sending $K_1 := \ker([\ ,\ ]\colon \bigwedge^2 X \to \mathrm{gr}^2(G)\otimes\Bbbk)$ onto $K_2 := \ker([\ ,\ ]\colon \bigwedge^2 X \to \mathfrak{H}^2)$.

As noted in the proof of Lemma 2.9, $K_1 = Y$; on the other hand, $Y = K_2$, by (2.1). Hence, $\phi$ extends to a graded Lie algebra automorphism $\phi\colon \mathfrak{H} \to \mathfrak{H}$. Upon completion, we obtain a group automorphism $\widehat{\phi}\colon \mathsf{exp}(\widehat{\mathfrak{H}}) \to \mathsf{exp}(\widehat{\mathfrak{H}})$. The desired normalized Malcev completion is $\kappa = \widehat{\phi}^{-1}\circ\kappa_G$. □

## 3. Alexander invariants of 1-formal groups

Our goal in this section is to derive a relation between the Alexander invariant and the holonomy Lie algebra of a finitely presented, 1-formal group.

3.1. **Alexander invariants.** Let $G$ be a group. Consider the exact sequence

$$0 \longrightarrow G'/G'' \xrightarrow{\ j\ } G/G'' \xrightarrow{\ p\ } G_{\mathrm{ab}} \longrightarrow 0\ , \tag{3.1}$$

where $G' = (G, G)$, $G'' = (G', G')$ and $G_{\mathrm{ab}} = G/G'$. Conjugation in $G/G''$ naturally makes $G'/G''$ into a module over the group ring $\mathbb{Z}G_{\mathrm{ab}}$. We call this module,

$$B_G = G'/G'',$$

the *Alexander invariant* of $G$. If $G = \pi_1(M)$, where $M$ is a connected CW-complex, one has the following useful topological interpretation for the Alexander invariant. Let $M' \to M$ be the Galois cover corresponding to $G' \subset G$. Then $B_G \otimes \Bbbk = H_1(M', \Bbbk)$, and the action of $G_{\mathrm{ab}}$ corresponds to the action in homology of the group of covering transformations. See [30, 16, 49] as classical references, and [13, 21] for a detailed treatment in the case of complements of hyperplane arrangements and hypersurfaces, respectively.

Now assume the group $G$ is finitely presented. Then $B_G \otimes \Bbbk$ is a finitely generated module over the Noetherian ring $\Bbbk G_{\mathrm{ab}}$. Denote by $I \subset \Bbbk G_{\mathrm{ab}}$ the augmentation ideal and set $X := G_{\mathrm{ab}} \otimes \Bbbk$. The $I$-adic completion $\widehat{B_G \otimes \Bbbk}$ is a finitely generated module over $\widehat{\Bbbk G_{\mathrm{ab}}} = \Bbbk[[X]]$, the formal power series ring on $X$. Note that $\Bbbk[[X]]$ is also the $(X)$-adic completion of the polynomial ring $\Bbbk[X]$.

Another invariant associated to a finitely presented group $G$ is the *infinitesimal Alexander invariant*, $B_{\mathfrak{H}(G)}$. By definition, this is the finitely generated module over $\Bbbk[X]$ with presentation matrix

$$\nabla := \delta_3 + \mathrm{id} \otimes \partial_G \colon \Bbbk[X] \otimes \Big( \bigwedge^3 X \oplus Y \Big) \to \Bbbk[X] \otimes \bigwedge^2 X, \tag{3.2}$$

where $Y = H_2(G, \Bbbk)$ and $\delta_3(x \wedge y \wedge z) = x \otimes y \wedge z - y \otimes x \wedge z + z \otimes x \wedge y$; see [60, Theorem 6.2], and also [14, 51] for related definitions. As the notation indicates, the module $B_{\mathfrak{H}(G)}$ depends only on the holonomy Lie algebra of $G$.

3.2. **The exponential base change.** Set $\mathfrak{H} = \mathfrak{H}(G)$ and denote by $\mathfrak{H}'$ (resp. $\mathfrak{H}''$) the first (resp. the second) derived Lie subalgebras. Consider the exact sequence of graded Lie algebras

$$0 \longrightarrow \mathfrak{H}'/\mathfrak{H}'' \xrightarrow{\ \iota\ } \mathfrak{H}/\mathfrak{H}'' \xrightarrow{\ \pi\ } \mathfrak{H}_{\mathrm{ab}} \longrightarrow 0\ , \tag{3.3}$$

where $\mathfrak{H}_{\mathrm{ab}} = \mathfrak{H}/\mathfrak{H}' = X$. Note that the adjoint representation of $\mathfrak{H}/\mathfrak{H}''$ induces a natural graded module structure on $\mathfrak{H}'/\mathfrak{H}''$, over the universal enveloping algebra $U(\mathfrak{H}_{\mathrm{ab}}) = \Bbbk[X]$.

Let $\kappa \colon G \to \mathsf{exp}(\widehat{\mathfrak{H}})$ be a normalized Malcev completion as in Lemma 2.10. Theorem 3.5 from [60] guarantees the factorization of $\kappa$ through Malcev completion homomorphisms $\kappa'' \colon G/G'' \to \mathsf{exp}(\widehat{\mathfrak{H}/\mathfrak{H}''})$ and $\kappa' \colon G_{\mathrm{ab}} \to \mathsf{exp}(\widehat{\mathfrak{H}_{\mathrm{ab}}})$. From the definitions, $\kappa' \circ p = \widehat{\pi} \circ \kappa''$. Furthermore, the completion of the exact sequence (3.3) with respect to degree filtrations is still exact. Therefore, there is an induced group

morphism $\kappa_0\colon G'/G'' \to \mathsf{exp}(\widehat{\mathfrak{H}'/\mathfrak{H}''})$ that fits into the following commuting diagram

$$\begin{tikzcd}
0 \arrow[r] & G'/G'' \arrow[r, "j"] \arrow[d, "\kappa_0"] & G/G'' \arrow[r, "p"] \arrow[d, "\kappa''"] & G_{\mathrm{ab}} \arrow[r] \arrow[d, "\kappa'"] & 0 \\
0 \arrow[r] & \mathsf{exp}(\widehat{\mathfrak{H}'/\mathfrak{H}''}) \arrow[r, "\widehat{\iota}"] & \mathsf{exp}(\widehat{\mathfrak{H}/\mathfrak{H}''}) \arrow[r, "\widehat{\pi}"] & \mathsf{exp}(\widehat{\mathfrak{H}}_{\mathrm{ab}}) \arrow[r] & 0
\end{tikzcd} \tag{3.4}$$

Note that $\mathfrak{H}^*_{\mathrm{ab}} = \mathfrak{H}^1_{\mathrm{ab}} = X$ has the structure of an abelian Lie algebra. Therefore, by the Campbell-Hausdorff formula (2.4), the exponential group structure on $\mathsf{exp}(\widehat{\mathfrak{H}}_{\mathrm{ab}}) = X$ coincides with the underlying abelian group structure of the $\Bbbk$-vector space $X$. Moreover, by the normalization property, $\kappa'$ coincides with the canonical map $G_{\mathrm{ab}} \to X$, $a \mapsto a \otimes 1$. Note also that the Lie algebra $\mathfrak{H}'/\mathfrak{H}''$ is abelian, hence $\mathsf{exp}(\widehat{\mathfrak{H}'/\mathfrak{H}''})$ is the underlying abelian group structure of the $\Bbbk$-vector space $\widehat{\mathfrak{H}'/\mathfrak{H}''}$, again by (2.4).

Now consider the morphism of $\Bbbk$-algebras

$$\exp\colon \Bbbk G_{\mathrm{ab}} \to \Bbbk[[X]], \tag{3.5}$$

given by $\exp(a) = e^{a\otimes 1}$, for $a \in G_{\mathrm{ab}}$. By the universality property of completions, there is an isomorphism of filtered $\Bbbk$-algebras, $\widehat{\exp}\colon \widehat{\Bbbk G_{\mathrm{ab}}} \xrightarrow{\simeq} \Bbbk[[X]]$, such that the morphism exp decomposes as $\Bbbk G_{\mathrm{ab}} \to \widehat{\Bbbk G_{\mathrm{ab}}} \xrightarrow{\widehat{\exp}} \Bbbk[[X]]$, where the first arrow is the natural map to the $I$-adic completion. Note that the completion $\widehat{\mathfrak{H}'/\mathfrak{H}''}$ has a canonical module structure over the $(X)$-adic completion $\widehat{\Bbbk[X]} = \Bbbk[[X]]$.

**Lemma 3.3.** *The map $\kappa_0 \otimes \Bbbk\colon (G'/G'') \otimes \Bbbk \to \widehat{\mathfrak{H}'/\mathfrak{H}''}$ is* exp*-linear, that is,*

$$(\kappa_0 \otimes \Bbbk)(\alpha \cdot \beta) = \exp(\alpha) \cdot (\kappa_0 \otimes \Bbbk)(\beta)\,,$$

*for $\alpha \in \Bbbk G_{\mathrm{ab}}$ and $\beta \in (G'/G'') \otimes \Bbbk$.*

*Proof.* It is enough to show that $\kappa''(a \cdot j(b) \cdot a^{-1}) = e^{p(a)\otimes 1} \cdot \kappa''(j(b))$ for $a \in G/G''$ and $b \in G'/G''$. To check this equality, recall the well-known conjugation formula in exponential groups (see Lazard [41]), which in our situation says

$$xyx^{-1} = \exp(\mathrm{ad}_x)(y),$$

for $x, y \in \mathsf{exp}(\widehat{\mathfrak{H}/\mathfrak{H}''})$. Hence $\kappa''(a \cdot j(b) \cdot a^{-1}) = e^{\mathrm{ad}_{\kappa''(a)}}(\kappa''(j(b)))$, which equals $e^{p(a)\otimes 1} \cdot \kappa''(j(b))$, since $\widehat{\pi} \circ \kappa''(a) = \kappa' \circ p(a) = p(a) \otimes 1$. ∎

3.4. **Completion of the Alexander invariant.** Recall that $B_G \otimes \Bbbk$ is a module over $\Bbbk G_{\mathrm{ab}}$, and $\widehat{\mathfrak{H}'/\mathfrak{H}''}$ is a module over $\widehat{\Bbbk[X]} = \Bbbk[[X]]$. Shift the canonical degree filtration on $\widehat{\mathfrak{H}'/\mathfrak{H}''}$ by setting $F'_q\,\widehat{\mathfrak{H}'/\mathfrak{H}''} := F_{q+2}\,\widehat{\mathfrak{H}'/\mathfrak{H}''}$, for each $q \geq 0$.

**Lemma 3.5.** *The $\Bbbk$-linear map $\kappa_0 \otimes \Bbbk\colon B_G \otimes \Bbbk \to \widehat{\mathfrak{H}'/\mathfrak{H}''}$ induces a filtered $\widehat{\exp}$-linear isomorphism between $\widehat{B_G \otimes \Bbbk}$, endowed with the filtration coming from the $I$-adic completion, and $\widehat{\mathfrak{H}'/\mathfrak{H}''}$, endowed with the shifted degree filtration $F'$.*

*Proof.* We start by proving that

$$\kappa_0 \otimes \Bbbk\,(I^q B_G \otimes \Bbbk) \subset F'_q\,\widehat{\mathfrak{H}'/\mathfrak{H}''} \tag{3.6}$$

for all $q \geq 0$. First note that $\widehat{\mathfrak{H}'/\mathfrak{H}''} = F_2\widehat{\mathfrak{H}'/\mathfrak{H}''}$, since $\mathfrak{H}'$ consists of elements of degree at least 2. Next, recall that $\exp(I) \subset (X)$. Finally, note that $(X)^r F_s\,\widehat{\mathfrak{H}'/\mathfrak{H}''} \subset F_{r+s}\,\widehat{\mathfrak{H}'/\mathfrak{H}''}$ for all $r, s \geq 0$. These observations, together with the exp-equivariance property from Lemma 3.3, establish the claim.

In view of (3.6), $\kappa_0 \otimes \Bbbk$ induces a filtered, $\widehat{\exp}$-linear map from $\widehat{B_G \otimes \Bbbk}$ to $\widehat{\mathfrak{H}'/\mathfrak{H}''}$. We are left with checking this map is a filtered isomorphism. For that, it is enough to show

$$\operatorname{gr}^q(\kappa_0 \otimes \Bbbk)\colon\ \operatorname{gr}^q_I(B_G \otimes \Bbbk) \to (\mathfrak{H}'/\mathfrak{H}'')^{q+2} \tag{3.7}$$

is an isomorphism, for each $q \geq 0$.

Recall from (3.4) that $\kappa'' \circ j = \widehat{\iota} \circ \kappa_0$. By a result of W. Massey [49, pp. 400–401], the map $j\colon G'/G'' \to G/G''$ induces isomorphisms

$$\operatorname{gr}^q(j) \otimes \Bbbk\colon\ \operatorname{gr}^q_I(B_G \otimes \Bbbk) \xrightarrow{\simeq} \operatorname{gr}^{q+2}(G/G'') \otimes \Bbbk \tag{3.8}$$

for all $q \geq 0$. Since $\kappa''$ is a Malcev completion, it induces isomorphisms

$$\operatorname{gr}^q(\kappa'')\colon\ \operatorname{gr}^q(G/G'') \otimes \Bbbk \xrightarrow{\simeq} (\mathfrak{H}/\mathfrak{H}'')^q \tag{3.9}$$

for all $q \geq 1$. Finally (and evidently), the inclusion map $\iota\colon \mathfrak{H}'/\mathfrak{H}'' \to \mathfrak{H}/\mathfrak{H}''$ induces isomorphisms $\operatorname{gr}^q(\iota)$, and thus $\operatorname{gr}^q(\widehat{\iota})$, for all $q \geq 2$. This finishes the proof. ◻

**Lemma 3.6.** *The $\Bbbk[[X]]$-module $\widehat{B_{\mathfrak{H}(G)}}$ is filtered isomorphic to the module $\widehat{\mathfrak{H}'/\mathfrak{H}''}$, endowed with the shifted filtration $F'$.*

*Proof.* Assign degree $q$ to $\bigwedge^q X$ and degree 2 to $Y$ in (3.2). Then $\mathfrak{H}'/\mathfrak{H}''$, viewed as a graded $\Bbbk[X]$-module via the exact sequence (3.3), is graded isomorphic to the $\Bbbk[X]$-module $\operatorname{coker}(\nabla)$, see [60, Theorem 6.2]. Taking $(X)$-adic completions, the claim follows. ◻

Putting Lemmas 3.5 and 3.6 together, we obtain the main result of this section.

**Theorem 3.7.** *Let $G$ be a 1-formal group. Then the $I$-adic completion of the Alexander invariant, $\widehat{B_G \otimes \Bbbk}$, is isomorphic to the $(X)$-adic completion of the infinitesimal Alexander invariant, $\widehat{B_{\mathfrak{H}(G)}}$, by a filtered $\widehat{\exp}$-linear isomorphism.*

3.8. **Fitting ideals.** We now recall some basic material from [26, §20.2]. Let $R$ be a commutative Noetherian ring, and let $N$ be a finitely generated $R$-module. Then $N$ admits a presentation of the form

$$R^r \xrightarrow{\nabla} R^s \longrightarrow N \longrightarrow 0 . \tag{3.10}$$

Define the $k$-th Fitting ideal of $N$ as follows: $\mathcal{F}_k(N)$ is the ideal generated by the $(s-k)$-minors of $\nabla$, if $0 < s-k \leq \min\{r,s\}$, $\mathcal{F}_k(N) = R$ if $s-k \leq 0$, and $\mathcal{F}_k(N) = 0$, otherwise. The Fitting ideals form an ascending chain, independent of the choice of presentation for $N$. Their construction commutes with base change.

If $R = \mathbb{C}[A]$ is the coordinate ring of an affine variety, we may consider the decreasing filtration of $A$ by the *Fitting loci* of $N$,

$$\mathcal{W}_k(N) := Z(\mathcal{F}_{k-1}(N)) \subset A. \tag{3.11}$$

These loci (which depend only on the coherent sheaf $\widetilde{N}$) define a decreasing filtration of $A$ by closed reduced subvarieties.

The next (elementary) lemma will be useful in the sequel.

**Lemma 3.9.** *Let $A$ be an affine variety, with coordinate ring $R$. For a point $t \in A$, denote by $\mathfrak{m}_t \subset R$ the corresponding maximal ideal. Let $N$ be a finitely generated $R$-module. Then*

$$t \in \mathcal{W}_k(N) \iff \dim_{\mathbb{C}}(N/\mathfrak{m}_t N) \geq k.$$

*Proof.* First note that $N/\mathfrak{m}_t N = R/\mathfrak{m}_t \otimes_R N$. Assuming $0 < s-k+1 \leq \min\{r,s\}$ in presentation (3.10), we may translate definition (3.11) to

$$t \in \mathcal{W}_k(N) \iff \operatorname{rank}_{\mathbb{C}}(R/\mathfrak{m}_t \otimes_R \nabla) \leq s-k. \tag{3.12}$$

We also have

$$\operatorname{rank}_{\mathbb{C}}(R/\mathfrak{m}_t \otimes_R \nabla) \leq s-k \iff \dim_{\mathbb{C}}(R/\mathfrak{m}_t \otimes_R N) \geq k. \tag{3.13}$$

These two equivalences yield the claim in this case. If $s-k+1 > r$, then $\mathcal{W}_k(N) = A$ by definition. In this situation, we also have $\dim_{\mathbb{C}}(R/\mathfrak{m}_t \otimes_R N) \geq s-r \geq k$. The remaining cases are similar. □

3.10. **Fitting loci and 1-formality.** Let $G$ be a finitely presented group, with Alexander invariant $B_G$ and infinitesimal Alexander invariant $B_{\mathfrak{H}(G)}$. Use the isomorphism $\widehat{\exp}\colon \widehat{\mathbb{C}G_{\mathrm{ab}}} \xrightarrow{\simeq} \mathbb{C}[[X]]$ to identify $\widehat{\mathbb{C}G_{\mathrm{ab}}}$, the $I$-adic completion of the group algebra $\mathbb{C}G_{\mathrm{ab}}$, with $\mathbb{C}[[X]]$, the $(X)$-adic completion of the polynomial ring $\mathbb{C}[X]$.

**Lemma 3.11.** *If $G$ is 1-formal, then*

$$\widehat{\exp}\colon \mathcal{F}_k(B_G \otimes \mathbb{C})\, \mathbb{C}[[X]] \xrightarrow{\simeq} \mathcal{F}_k(B_{\mathfrak{H}(G)})\, \mathbb{C}[[X]],$$

*for all $k \geq 0$.*

*Proof.* This is a direct consequence of Theorem 3.7, by base change. □

Let $T_1\mathbb{T}_G = \mathrm{Hom}(G, \mathbb{C})$ be the tangent space at the origin 1 to the complex analytic torus $\mathbb{T}_G = \mathrm{Hom}(G, \mathbb{C}^*)$. Then $T_1\mathbb{T}_G$ is the Lie algebra of $\mathbb{T}_G$, and the exponential map $\mathbb{C} \to \mathbb{C}^*$, $a \mapsto e^a$, induces a local analytic isomorphism

$$\exp\colon (T_1\mathbb{T}_G, 0) \xrightarrow{\simeq} (\mathbb{T}_G, 1). \tag{3.14}$$

**Proposition 3.12.** *If $G$ is 1-formal, then the exponential map induces an isomorphism of analytic germs*

$$\exp\colon (\mathcal{W}_k(B_{\mathfrak{H}(G)}), 0) \xrightarrow{\simeq} (\mathcal{W}_k(B_G \otimes \mathbb{C}), 1),$$

*for all $k \geq 0$.*

*Proof.* In geometric terms, the $I$-adic completion map can be written as the composite $\mathbb{C}G_{\mathrm{ab}} \to \mathbb{C}\{X\} \to \mathbb{C}[[X]]$, where the first arrow takes global regular functions on $\mathbb{T}_G$ to their analytic germs at 1, and the second arrow is Taylor expansion. One has a similar decomposition for the $(X)$-adic completion, $\mathbb{C}[X] \to \mathbb{C}\{X\} \to \mathbb{C}[[X]]$, obtained by taking germs (respectively, Taylor expansions) at 0.

With these identifications, the map $\exp\colon \mathbb{C}G_{\mathrm{ab}} \to \mathbb{C}[[X]]$ from (3.5) can be viewed as the composite $\mathbb{C}G_{\mathrm{ab}} \to \mathbb{C}\{X\} \to \mathbb{C}\{X\} \to \mathbb{C}[[X]]$, where the middle arrow is the map induced by (3.14) on local coordinate rings.

Since $\mathbb{C}[[X]]$ is faithfully flat over $\mathbb{C}\{X\}$, see [71, p. 36], we obtain from Lemma 3.11 an exponential identification

$$\exp\colon \mathcal{F}_k(B_G \otimes \mathbb{C})\, \mathbb{C}\{X\} \xrightarrow{\simeq} \mathcal{F}_k(B_{\mathfrak{H}(G)})\, \mathbb{C}\{X\}$$

for all $k \geq 0$. The claim follows by taking the corresponding germs of zero sets. □

## 4. Germs of cohomology support loci

In this section, we study the cohomology support loci of a finitely presented group $G$: the resonance varieties $\mathcal{R}_k(G)$ and the characteristic varieties $\mathcal{V}_k(G)$. Under a 1-formality assumption on $G$, we show that the exponential map restricts to an isomorphism of analytic germs between $(\mathcal{R}_k(G), 0)$ and $(\mathcal{V}_k(G), 1)$. We work over the field $\Bbbk = \mathbb{C}$, unless otherwise specified.

4.1. **Resonance varieties.** Let $H^1$ and $H^2$ be two finite-dimensional complex vector spaces and let $\mu\colon H^1 \wedge H^1 \to H^2$ be a linear map. Set $H^0 = \mathbb{C}$. For an element $z \in H^1$, denote by $\cdot z$ right-multiplication by $z$ in the exterior algebra $\bigwedge^* H^1$, and by $\mu_z$ the composite $H^1 \xrightarrow{\cdot z} H^1 \wedge H^1 \xrightarrow{\mu} H^2$. We then have a cochain complex

$$(H^\bullet, \mu_z)\colon\ 0 \longrightarrow H^0 \xrightarrow{\cdot z} H^1 \xrightarrow{\mu_z} H^2 \longrightarrow 0\,, \tag{4.1}$$

with cohomology denoted by $H^*(H^\bullet, \mu_z)$. For each $k \geq 0$, the $k$-th *resonance variety* of $\mu$ is defined by

$$\mathcal{R}_k(\mu) := \{z \in H^1 \mid \dim H^1(H^\bullet, \mu_z) \geq k\}\,. \tag{4.2}$$

Clearly, the jumping loci $\{\mathcal{R}_k(\mu)\}_k$ form a decreasing filtration of $H^1$ by closed homogeneous subvarieties, starting from $\mathcal{R}_0(\mu) = H^1$.

For a CW-complex $M$ as above, take $H^i = H^i(M, \mathbb{C})$, take $\mu$ to be the cup-product map $\cup_M$, and define the corresponding resonance varieties $\mathcal{R}_k(M) := \mathcal{R}_k(\cup_M)$. Similarly, for a finitely presented group $G$, define

$$\mathcal{R}_k(G) := \mathcal{R}_k(K(G, 1)). \tag{4.3}$$

Obviously, the resonance varieties (4.2) depend only on the corestriction of $\mu$ to its image. By applying this remark to the classifying map $f\colon M \to K(\pi_1(M), 1)$, which induces an isomorphism on $H^1$ and a monomorphism on $H^2$, we see that

$$\mathcal{R}_k(M) = \mathcal{R}_k(\pi_1(M)) \tag{4.4}$$

for all $k \geq 0$.

**Lemma 4.2.** *Let $G$ be a finitely presented group. Then the equality*

$$\mathcal{W}_k(B_{\mathfrak{H}(G)}) \setminus \{0\} = \mathcal{R}_k(G) \setminus \{0\}$$

*holds for all $k \geq 0$.*

*Proof.* Pick any $z \in H^1(G, \mathbb{C}) \setminus \{0\}$. By applying Lemma 3.9 to $B_{\mathfrak{H}(G)}$, we infer that $z \in \mathcal{W}_k(B_{\mathfrak{H}(G)})$ if and only if $\dim_{\mathbb{C}} \operatorname{coker}(\nabla(z)) \geq k$. Note that $H^1(G, \mathbb{C})$ equals the dual vector space ${}^{\sharp}X$ of $X$. Consider the cochain complex $(\bigwedge^{\bullet} {}^{\sharp}X, \lambda_z)$, where $\lambda_z$ denotes left multiplication by $z$. Since $z \neq 0$, this complex is obviously exact. Let $(\bigwedge^{\bullet} X, {}^{\sharp}\lambda_z)$ be the dual chain complex, which is again exact. It is straightforward to check that the restriction of ${}^{\sharp}\lambda_z$ to $\bigwedge^3 X$ equals $\delta_3(z)$, with $\delta_3$ as in the presentation (3.2) of $B_{\mathfrak{H}(G)}$.

Denoting by $\delta_2(z)$ the restriction of ${}^{\sharp}\lambda_z$ to $\bigwedge^2 X$, use the exactness of the complex $(\bigwedge^{\bullet} X, {}^{\sharp}\lambda_z)$ to obtain the following isomorphism:

$$\operatorname{coker}(\nabla(z)) \cong \operatorname{im}(\delta_2(z))/\operatorname{im}(\delta_2(z) \circ \partial_G). \tag{4.5}$$

By exactness again, $\dim_{\mathbb{C}} \operatorname{im}(\delta_2(z)) = n - 1$. Hence $z \in \mathcal{W}_k(B_{\mathfrak{H}(G)})$ if and only if $\operatorname{rank}(\delta_2(z) \circ \partial_G) \leq n - 1 - k$. Using (2.1)–(2.2) and (4.1)–(4.3), we see that the linear map dual to $\delta_2(z) \circ \partial_G$ is $-\mu_z$. Consequently, $z \in \mathcal{W}_k(B_{\mathfrak{H}(G)})$ if and only if $\operatorname{rank}(\mu_z) \leq n - 1 - k$, that is, if and only if $z \in \mathcal{R}_k(G)$, see (4.2)–(4.3). $\square$

4.3. **Characteristic varieties.** Let $M$ be a connected CW-complex with finite 2-skeleton. Set $G = \pi_1(M)$ and consider the character torus $\mathbb{T}_G = \operatorname{Hom}(G, \mathbb{C}^*)$. This is an algebraic group of the form $(\mathbb{C}^*)^n \times F$, where $n = b_1(M)$ and $F$ is a finite abelian group. Identifying the point $\rho \in \mathbb{T}_G$ with a rank one local system ${}_{\rho}\mathbb{C}$ on $M$ (that is, a left one-dimensional $\mathbb{C}$–representation of $\pi_1(M)$), one may define the $k$-th *characteristic variety* for all $k \geq 0$ by

$$\mathcal{V}_k(M) := \{\rho \in \mathbb{T}_G \mid \dim H^1(M, {}_{\rho}\mathbb{C}) \geq k\}. \tag{4.6}$$

Here $H^\bullet(M, {}_\rho\mathbb{C})$ denotes twisted cohomology, see e.g. [73, Chapter VI]. Viewing $\rho$ as a right representation, it follows by duality that we may replace $H^1(M, {}_\rho\mathbb{C})$ by twisted homology, $H_1(M, {}_\rho\mathbb{C})$, in definition (4.6).

Plainly, $\{\mathcal{V}_k(M)\}_k$ is a decreasing filtration of the torus $\mathbb{T}_G$ by closed algebraic subvarieties, starting from $\mathcal{V}_0(M) = \mathbb{T}_G$. Describing this filtration is equivalent to understanding the degree one (co)homology of $M$ with coefficients in an arbitrary rank one local system, a very difficult task in general.

For a finitely presented group $G$, define

$$\mathcal{V}_k(G) := \mathcal{V}_k(K(G,1)). \tag{4.7}$$

If $M$ is a $CW$-complex as above, with $G = \pi_1(M)$, then an Eilenberg-MacLane space $K(G,1)$ can be obtained from $M$ by attaching cells of dimension $\geq 3$; hence

$$\mathcal{V}_k(M) = \mathcal{V}_k(\pi_1(M)), \tag{4.8}$$

for all $k \geq 0$.

Let $G = \langle x_1, \dots, x_s \mid w_1, \dots, w_r \rangle$ be a finite presentation for $G$. The 2-complex $M$ associated to this presentation has $s$ one-cells, corresponding to the generators $x_i$, and $r$ two-cells, attached according to the defining relations $w_j$. Denote by

$$\widetilde{C}_\bullet = C_\bullet(\widetilde{M})\colon\ 0 \longrightarrow (\mathbb{Z}G)^r \xrightarrow{d_2} (\mathbb{Z}G)^s \xrightarrow{d_1} \mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z} \tag{4.9}$$

the augmented $G$-equivariant cellular chain complex of the universal cover $\widetilde{M}$.

**Remark 4.4.** If we tensor the chain complex (4.9) (where the last term $\mathbb{Z}$ is replaced by 0) with $\mathbb{C}$, via the ring extension $\mathbb{Z}G \to \mathbb{C}G_{\mathrm{ab}} \xrightarrow{\rho} \mathbb{C}$ associated to a character $\rho$, we obtain a complex of finite-dimensional $\mathbb{C}$-vector spaces, namely

$$0 \longrightarrow \mathbb{C}^r \xrightarrow{d_2(\rho)} \mathbb{C}^s \xrightarrow{d_1(\rho)} \mathbb{C} \longrightarrow 0\ .$$

Here the differentials $d_k(\rho)$ are represented by matrices whose entries are regular functions in $\rho \in \mathbb{T}_G$. It follows that the function

$$\dim H_1(G, {}_\rho\mathbb{C}) = \dim\ker d_1(\rho) - \operatorname{rank} d_2(\rho) = s - \operatorname{rank} d_1(\rho) - \operatorname{rank} d_2(\rho)$$

is lower semi-continuous with respect to $\rho$, i.e.,

$$\dim H_1(G, {}_\rho\mathbb{C}) \leq \dim H_1(G, {}_{\rho_0}\mathbb{C})$$

for $\rho$ in a neighborhood of a fixed character $\rho_0$.

Let $\mathbb{Z}G \to \mathbb{C}G_{\mathrm{ab}}$ be the base change associated to the abelianization morphism. As mentioned in §3, there is a natural $\mathbb{C}G_{\mathrm{ab}}$-module identification

$$B_G \otimes \mathbb{C} = H_1(\mathbb{C}G_{\mathrm{ab}} \otimes_{\mathbb{Z}G} \widetilde{C}_\bullet). \tag{4.10}$$

Let $\mathbb{Z}G \to \mathbb{C}G_{\mathrm{ab}} \xrightarrow{\rho} \mathbb{C}$ be the change of rings corresponding to a character $\rho \in \mathbb{T}_G$. As follows from (4.8),

$$\rho \in \mathcal{V}_k(G) \iff \dim_{\mathbb{C}} H_1({}_\rho\mathbb{C} \otimes_{\mathbb{Z}G} \widetilde{C}_\bullet) \geq k. \tag{4.11}$$

By applying Lemma 3.9 to $N = B_G \otimes \mathbb{C}$ and using (4.10), we infer that

$$\rho \in \mathcal{W}_k(B_G \otimes \mathbb{C}) \iff \dim_{\mathbb{C}}({}_\rho\mathbb{C} \otimes_{\mathbb{C}G_{\mathrm{ab}}} H_1 K_\bullet) \geq k, \tag{4.12}$$

where $K_\bullet$ denotes the $\mathbb{C}G_{\mathrm{ab}}$-chain complex $\mathbb{C}G_{\mathrm{ab}} \otimes_{\mathbb{Z}G} \widetilde{C}_\bullet$.

With these preliminaries, we may state the following analogue of Lemma 4.2.

**Lemma 4.5.** *Let $G$ be a finitely presented group. Then the equality*

$$\mathcal{W}_k(B_G \otimes \mathbb{C}) \setminus \{1\} = \mathcal{V}_k(G) \setminus \{1\}$$

*holds for all $k \geq 0$.*

*Proof.* Let $R = \mathbb{C}G_{\mathrm{ab}}$. By (4.11) and (4.12), it is enough to show $H_1({}_\rho\mathbb{C} \otimes_R K_\bullet) = {}_\rho\mathbb{C} \otimes_R H_1 K_\bullet$, for $\rho \in \mathbb{T}_G \setminus \{1\}$. An analysis of the spectral sequence associated to the free, finite chain complex $K_\bullet$ over the ring $R$, and the base change $\rho\colon R \to \mathbb{C}$,

$$E^2_{s,t} = \mathrm{Tor}^R_s({}_\rho\mathbb{C}, H_t K_\bullet) \Rightarrow H_{s+t}({}_\rho\mathbb{C} \otimes_R K_\bullet),$$

see [46, Theorem XII.12.1], shows we only need to check $\mathrm{Tor}^R_*({}_\rho\mathbb{C}, {}_\epsilon\mathbb{C}) = 0$, where $\epsilon\colon R \to \mathbb{C}$ is the augmentation map.

Denote by $F \subset G_{\mathrm{ab}}$ the torsion subgroup, and by $G_{\mathrm{ab}}/F = \mathbb{Z}^n$ the torsion-free part of $G_{\mathrm{ab}}$. Since $R \cong \mathbb{C}\mathbb{Z}^n \otimes \mathbb{C}F$, it follows from the Künneth formula that

$$\mathrm{Tor}^R_*({}_\rho\mathbb{C}, {}_\epsilon\mathbb{C}) = \mathrm{Tor}^{\mathbb{C}\mathbb{Z}^n}_*({}_{\rho'}\mathbb{C}, {}_\epsilon\mathbb{C}) \otimes \mathrm{Tor}^{\mathbb{C}F}_*({}_{\rho''}\mathbb{C}, {}_\epsilon\mathbb{C}),$$

where $\rho = (\rho', \rho'') \in \mathbb{T}_{\mathbb{Z}^n} \times \mathbb{T}_F$. It is well-known that $\mathrm{Tor}^{\mathbb{C}\mathbb{Z}^n}_*({}_{\rho'}\mathbb{C}, {}_\epsilon\mathbb{C}) = 0$, if $\rho' \neq 1$, see e.g. [35, pp. 215–216]. Thus, it is enough to show $\mathrm{Tor}^{\mathbb{C}F}_*({}_{\rho''}\mathbb{C}, {}_\epsilon\mathbb{C}) = 0$, if $\rho'' \neq 1$, for any finite cyclic group $F$. This can be easily checked, using the standard periodic resolution of ${}_\epsilon\mathbb{C}$ over $\mathbb{C}F$, see e.g. [46, IV.7]. $\square$

4.6. **Matching analytic germs.** The next theorem proves the first part of Theorem A from the Introduction. It implies, in particular, that the twisted (degree one) cohomology with rank one local systems is computable, near the trivial character 1 and under a formality assumption, only in terms of the cup-product map on $H^1$.

**Theorem 4.7.** *Let $G$ be a 1-formal group. Then the exponential map from $(T_1\mathbb{T}_G, 0)$ to $(\mathbb{T}_G, 1)$ restricts for each $k \geq 0$ to an isomorphism of analytic germs,*

$$\exp\colon (\mathcal{R}_k(G), 0) \xrightarrow{\simeq} (\mathcal{V}_k(G), 1).$$

*Proof.* We need to do two things: replace in Proposition 3.12 the germ $(\mathcal{W}_k(B_{\mathfrak{H}(G)}), 0)$ by $(\mathcal{R}_k(G), 0)$, and the germ $(\mathcal{W}_k(B_G \otimes \mathbb{C}), 1)$ by $(\mathcal{V}_k(G), 1)$.

Set $n = b_1(G)$. Then plainly $0 \in \mathcal{R}_k(G)$ and $1 \in \mathcal{V}_k(G)$ for $k \leq n$. It is also clear that $(\mathcal{R}_k(G), 0)$ is the germ of the empty set for $k > n$. To see that the same holds

for the germ $(\mathcal{V}_k(G), 1)$, note that $\dim_{\mathbb{C}} H^1(G, {}_\rho\mathbb{C}) \leq \dim_{\mathbb{C}} H^1(G, {}_1\mathbb{C}) = n$ for any $\rho$ near 1, by semi-continuity, see Remark 4.4. Therefore it is enough to make the two aforementioned replacements only away from 0 (resp. 1). This can be done using Lemmas 4.2 and 4.5. □

**Remark 4.8.** Suppose $M$ is the complement of a hyperplane arrangement in $\mathbb{C}^m$, with fundamental group $G = \pi_1(M)$. In this case, Theorem 4.7 can be deduced from results of Esnault–Schechtman–Viehweg [27] and Schechtman–Terao–Varchenko [64]. In fact, one can show that there is a combinatorially defined open neighborhood $U$ of 0 in $H^1(M, \mathbb{C})$ with the property that $H^*(M, {}_\rho\mathbb{C}) \cong H^*(H^\bullet(M, \mathbb{C}), \mu_z)$, for all $z \in U$, where $\rho = \exp(z)$. A similar approach works as soon as $W_1(H^1(M, \mathbb{C})) = 0$, in particular, for complements of arrangements of hypersurfaces in projective or affine space. For details, see [21, Corollary 4.6].

The local statement from Theorem 4.7 is the best one can hope for, as shown by the following classical example.

**Example 4.9.** Let $M$ be the complement in $S^3$ of a tame knot $K$. Since $M$ is a homology circle, it follows easily that $M$ is a formal space; therefore, its fundamental group, $G = \pi_1(M)$, is a 1-formal group. Let $\Delta(t) \in \mathbb{Z}[t, t^{-1}]$ be the Alexander polynomial of $K$. It is readily seen that $\mathcal{V}_1(G) = \{1\} \coprod \mathrm{Zero}(\Delta)$ and $\mathcal{R}_1(G) = \{0\}$. Thus, if $\Delta(t) \not\equiv 1$, then $\exp(\mathcal{R}_1(G)) \neq \mathcal{V}_1(G)$.

Even though the germ of $\mathcal{V}_1(G)$ at 1 provides no information in this case, the global structure of $\mathcal{V}_1(G)$ is quite meaningful. For example, if $K$ is an algebraic knot, then $\Delta(t)$ must be product of cyclotomic polynomials, as follows from work of Brauner and Zariski from the 1920s, see [54].

**Remark 4.10.** Let $M$ be the complement in $\mathbb{C}^2$ of an algebraic curve, with fundamental group $G = \pi_1(M)$, and let $\Delta(t) \in \mathbb{Z}[t, t^{-1}]$ be the Alexander polynomial of the total linking cover, as defined by Libgober; see [42] for details and references. It was shown in [42] that all the roots of $\Delta(t)$ are roots of unity. This gives restrictions on which finitely presented groups can be realized as fundamental groups of plane curve complements.

Let $\Delta^G \in \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ be the multivariable Alexander polynomial of an arbitrary quasi-projective group. Starting from Theorem B(2), we prove in [24] that $\Delta^G$ must have a single essential variable, if $n \neq 2$. Examples from [59] show that this new obstruction efficiently detects non-quasi-projectivity of (local) algebraic link groups. Note that all roots of the one-variable (local) Alexander polynomial $\Delta(t)$ of algebraic links are roots of unity; see [54].

4.11. **Initial ideals and tangent cones.** Let $(X, 0)$ be a reduced analytic space germ at the origin of $\mathbb{C}^n$ and let $I = I(X, 0) \subset \mathcal{O}_n$ be the ideal of analytic function germs at the origin of $\mathbb{C}^n$ vanishing on $(X, 0)$. Any non-zero $f \in \mathcal{O}_n$ can be uniquely written as a sum

$$f = f_m + f_{m+1} + \cdots ,$$

where each $f_k$ is a homogeneous polynomial of degree $k$, and $f_m \neq 0$. We call $f_m$ the initial form of the germ $f$ and denote it by $\mathrm{in}(f)$.

The *initial ideal* of $I$, denoted $\mathrm{in}(I)$, is the polynomial ideal spanned by all initial forms of non-zero elements of $I$. The *tangent cone* of the germ $(X, 0)$, denoted $TC_0(X)$, is the affine cone in $\mathbb{C}^n$ given by the zero-set of the initial ideal $\mathrm{in}(I)$. If $X$ is an algebraic subvariety in $\mathbb{C}^n$ and $p \in X$ is any point, then the tangent cone $TC_p(X)$ is defined in the obvious way, i.e., by translating the germ $(X, p)$ to the origin. For more geometric details on this definition see Whitney [74, pp. 210–228]; for relations to Gröbner bases, see Cox-Little-O'Shea [15].

Tangent cones enjoy the following functoriality property. Let $(X, 0) \subset (\mathbb{C}^n, 0)$ and $(Y, 0) \subset (\mathbb{C}^p, 0)$ be two reduced analytic space germs, and let $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be an analytic map germ such that $f(X, 0) \subset (Y, 0)$. Then the differential $d_0 f\colon T_0\mathbb{C}^n \to T_0\mathbb{C}^p$ satisfies $d_0 f(TC_0(X)) \subset TC_0(Y)$. In particular, if $f$ is a local analytic isomorphism, then its differential induces a linear isomorphism, $d_0 f\colon TC_0(X) \xrightarrow{\simeq} TC_0(Y)$.

4.12. **The tangent cone formula.** Using Theorem 4.7 and the above discussion, we obtain the following tangent cone formula, which generalizes results from [14, 43], and finishes the proof of Theorem A from the Introduction.

**Theorem 4.13.** *If the group $G$ is 1-formal, then $TC_1(\mathcal{V}_k(G)) = \mathcal{R}_k(G)$, for all $0 \leq k \leq b_1(G)$.*

**Remark 4.14.** In general, it is difficult to verify the 1-formality of a finitely presented group $G$ directly from Definition 2.4. The above tangent cone formula provides a new, computable obstruction to 1-formality. Indeed, recall the equivariant chain complex $\widetilde{C}_\bullet$ from (4.9). It follows from (4.11) that the characteristic varieties $\mathcal{V}_k(G)$ may be computed from the Fitting ideals of the $\mathbb{C}G_{\mathrm{ab}}$-module presented by the *Alexander matrix* associated to $\mathbb{C}G_{\mathrm{ab}} \otimes_{\mathbb{Z}G} d_2$.

It is a classical fact that the Alexander matrix is computable directly from a finite presentation for $G$, by means of the Fox calculus, see [30]. The passage from the variety $\mathcal{V}_k(G)$ to the tangent cone $TC_1(\mathcal{V}_k(G))$ is achieved by effective commutative algebra methods, as described in [15]; for details, see e.g. [13, 14].

It is even simpler to determine the resonance varieties $\mathcal{R}_k(G)$: compute the cup-product map $\cup_G$ and the holonomy Lie algebra $\mathfrak{H}(G)$ directly from the group presentation, by Fox calculus (or, equivalently, by Magnus expansion), and then use Fitting ideals to compute $\mathcal{R}_k(G)$ as in Lemma 4.2; for details, see e.g. [51].

## 5. Regular maps onto curves

In this section, we discuss the relationship between the cohomology jumping loci of a quasi-compact Kähler manifold $M$, holomorphic maps from $M$ to complex curves, and isotropic subspaces in $H^*(M, \mathbb{C})$. The basic tool is a result of Arapura [2], which we start by recalling.

5.1. **Arapura's theorem.** First, we establish some terminology. By a *curve* we mean a smooth, connected, complex algebraic variety of dimension 1. A curve $C$ admits a canonical compactification $\overline{C}$, obtained by adding a finite number of points.

Following [2, p. 590], we say a map $f\colon M \to C$ from a connected, quasi-compact Kähler manifold $M$ to a curve $C$ is *admissible* if $f$ is holomorphic and surjective, and has a holomorphic, surjective extension with connected fibers, $\overline{f}\colon \overline{M} \to \overline{C}$, where $\overline{M}$ is a smooth compactification, obtained by adding divisors with normal crossings.

With these preliminaries, we can state Arapura's result [2, Proposition V.1.7], in a slightly modified form, suitable for our purposes.

**Theorem 5.2.** *Let $M$ be a connected, quasi-compact Kähler manifold. Denote by $\{\mathcal{V}^\alpha\}_\alpha$ the set of irreducible components of $\mathcal{V}_1(\pi_1(M))$ containing $1$. If $\dim \mathcal{V}^\alpha > 0$, then the following hold.*

(1) *There is an admissible map, $f_\alpha\colon M \to C_\alpha$, where $C_\alpha$ is a smooth curve with $\chi(C_\alpha) < 0$, such that*
$$\mathcal{V}^\alpha = f_\alpha^* \mathbb{T}_{\pi_1(C_\alpha)}$$
*and $(f_\alpha)_\sharp\colon \pi_1(M) \to \pi_1(C_\alpha)$ is surjective.*

(2) *There is an isomorphism*
$$H^1(M, {}_{f_\alpha^*\rho}\mathbb{C}) \cong H^1(C_\alpha, {}_\rho\mathbb{C}),$$
*for all except finitely many local systems $\rho \in \mathbb{T}_{\pi_1(C_\alpha)}$.*

*Proof.* Part (1). Proposition V.1.7 from [2] guarantees the existence of an admissible map $f_\alpha\colon M \to C_\alpha$ to a curve $C_\alpha$ with $\chi(C_\alpha) < 0$ such that $\mathcal{V}^\alpha = f_\alpha^* \mathbb{T}_{\pi_1(C_\alpha)}$. The surjectivity of $(f_\alpha)_\sharp$ is implicit in Arapura's proof. It follows from two standard facts. First, the generic fibers of $f_\alpha$ are connected, as soon as this happens for $\overline{f}_\alpha$, and second, there is a Zariski open subset $U_\alpha \subset C_\alpha$ with the property that $f_\alpha\colon f_\alpha^{-1}(U_\alpha) \to U_\alpha$ is a locally trivial fibration.

Part (2). This is stated in the proof of Proposition V.1.7 from [2], with "all except finitely many" replaced by "infinitely many," but a careful look at Arapura's argument reveals that actually the finer property holds. □

When $M$ is compact, similar results to Arapura's were obtained previously by Beauville [4] and Simpson [67]. The closely related construction of regular mappings from an algebraic variety $M$ to a curve $C$ starting with suitable differential forms on $M$ goes back to Castelnuovo–de Franchis, see Catanese [9]. When both $M$ and $C$ are compact, the existence of a non-constant holomorphic map $M \to C$ is closely related to the existence of an epimorphism $\pi_1(M) \twoheadrightarrow \pi_1(C)$, see Beauville [3] and Green–Lazarsfeld [31]. In the non-compact case, this phenomenon is discussed in Corollary V.1.9 from [2].

**Corollary 5.3.** *Let $M$ be a connected, quasi-compact Kähler manifold, with fundamental group $G = \pi_1(M)$. Assume that $G$ is $1$-formal, with $b_1(G) > 0$ and*

$\mathcal{R}_1(G) \neq \{0\}$. *Then all irreducible components of* $\mathcal{V}_1(G)$ *containing* 1 *are positive-dimensional. Realize each irreducible component* $\mathcal{V}^\alpha$ *of* $\mathcal{V}_1(G)$ *containing* 1 *by pullback via* $f_\alpha\colon M \to C_\alpha$*, as in Theorem 5.2(1). Then* $T_1(\mathcal{V}^\alpha) = f_\alpha^* H^1(C_\alpha, \mathbb{C})$,

$$\mathcal{R}_1(G) = \bigcup\nolimits_\alpha f_\alpha^* H^1(C_\alpha, \mathbb{C}), \tag{5.1}$$

*and this decomposition coincides with the decomposition of* $\mathcal{R}_1(G)$ *into irreducible components,* $\mathcal{R}_1(G) = \bigcup_\alpha \mathcal{R}^\alpha$*. In particular,* $\dim \mathcal{R}^\alpha = b_1(C_\alpha)$*, for all* $\alpha$.

*Proof.* Since $b_1(G) > 0$ and $\mathcal{R}_1(G) \neq \{0\}$, all irreducible components $\mathcal{R}^\alpha$ of the algebraic set $\mathcal{R} = \mathcal{R}_1(G)$ are positive dimensional. Since $G$ is 1-formal, the tangent cone formula from Theorem 4.13 implies that all irreducible components $\mathcal{V}^\alpha$ of $\mathcal{V}_1(G)$ containing 1 are positive dimensional. Hence, Theorem 5.2(1) applies.

By Theorem 4.13, $\mathcal{R}_1(G) = TC_1(\mathcal{V}_1(G)) = \bigcup_\alpha TC_1(\mathcal{V}^\alpha)$. From Theorem 5.2(1), we know that each $\mathcal{V}^\alpha$ is an algebraic connected subtorus of $\mathbb{T}_G$, isomorphic to $\mathbb{T}_{\pi_1(C_\alpha)}$ via $f_\alpha^*$, since $(f_\alpha)_\sharp$ is onto. It follows from §4.11 that $TC_1(\mathcal{V}^\alpha) = T_1(\mathcal{V}^\alpha) = f_\alpha^* H^1(C_\alpha, \mathbb{C})$. This establishes (5.1). That this decomposition of $\mathcal{R}_1(G)$ coincides with the one into irreducible components follows from the fact that connected algebraic subtori are determined by their Lie algebras; see e.g. [37, 13.1]. □

5.4. **Isotropic subspaces.** Before proceeding, we introduce some notions which will be of considerable use in the sequel. Let $\mu\colon H^1 \wedge H^1 \to H^2$ be a $\mathbb{C}$-linear map, and $\mathcal{R}_k(\mu) \subset H^1$ be the corresponding resonance varieties, as defined in (4.2). One way to construct elements in these varieties is as follows.

**Lemma 5.5.** *Suppose* $V \subset H^1$ *is a linear subspace of dimension* $k$*. Set* $i = \dim \operatorname{im}(\mu\colon V \wedge V \to H^2)$*. If* $i < k-1$*, then* $V \subset \mathcal{R}_{k-i-1}(\mu) \subset \mathcal{R}_1(\mu)$.

*Proof.* Let $x \in V$, and set $x_V^\perp = \{y \in V \mid \mu(x \wedge y) = 0\}$. Clearly, $\dim x_V^\perp \geq k - i$. On the other hand, $x_V^\perp / \mathbb{C} \cdot x \subset H^1(H^\bullet, \mu_x)$, and so $x \in \mathcal{R}_{k-i-1}(\mu)$. □

Therefore, the subspaces $V \subset H^1$ for which $\dim \operatorname{im}(\mu\colon V \wedge V \to H^2)$ is small are particularly interesting. This remark gives a preliminary motivation for the following key definition.

**Definition 5.6.** Let $\mu\colon \bigwedge^2 H^1 \to H^2$ be a $\mathbb{C}$-linear mapping, and let $V \subset H^1$ be a $\mathbb{C}$-linear subspace.

(i) $V$ is 0-*isotropic* (or simply, *isotropic*) with respect to $\mu$ if the restriction $\mu^V\colon \bigwedge^2 V \to H^2$ is trivial.
(ii) $V$ is 1-*isotropic* with respect to $\mu$ if the restriction $\mu^V\colon \bigwedge^2 V \to H^2$ has 1-dimensional image and is a non-degenerate skew-symmetric bilinear form.

**Example 5.7.** Let $C$ be a smooth curve, and let $\mu_C\colon \bigwedge^2 H^1(C, \mathbb{C}) \to H^2(C, \mathbb{C})$ be the usual cup-product map. There are two cases of interest to us.

(i) If $C$ is not compact, then $H^2(C, \mathbb{C}) = 0$ and so any subspace $V \subset H^1(C, \mathbb{C})$ is isotropic.

(ii) If $C$ is compact, of genus $g \geq 1$, then $H^2(C,\mathbb{C}) = \mathbb{C}$ and $H^1(C,\mathbb{C})$ is 1-isotropic.

Now let $\mu_1\colon \bigwedge^2 H_1^1 \to H_1^2$ and $\mu_2\colon \bigwedge^2 H_2^1 \to H_2^2$ be two $\mathbb{C}$-linear maps.

**Definition 5.8.** The maps $\mu_1$ and $\mu_2$ are *equivalent* (notation $\mu_1 \simeq \mu_2$) if there exist linear isomorphisms $\phi^1\colon H_1^1 \to H_2^1$ and $\phi^2\colon \operatorname{im}(\mu_1) \to \operatorname{im}(\mu_2)$ such that $\phi^2 \circ \mu_1 = \mu_2 \circ \wedge^2 \phi^1$.

The key point of this definition is that the $k$-resonant varieties $\mathcal{R}_k(\mu_1)$ and $\mathcal{R}_k(\mu_2)$ can be identified under $\phi^1$ when $\mu_1 \simeq \mu_2$. Moreover, subspaces that are either 0-isotropic or 1-isotropic with respect to $\mu_1$ and $\mu_2$ are matched under $\phi^1$.

5.9. **Admissible maps and isotropic subspaces.** We now consider in more detail which admissible maps $f_\alpha\colon M \to C_\alpha$ may occur in Theorem 5.2.

**Proposition 5.10.** *Let $M$ be a connected quasi-compact Kähler manifold, and let $f\colon M \to C$ be an admissible map onto the smooth curve $C$.*

(1) *If $W_1(H^1(M,\mathbb{C})) = H^1(M,\mathbb{C})$, then the curve $C$ is either compact, or it is obtained from a compact smooth curve $\overline{C}$ by deleting a single point.*
(2) *If $W_1(H^1(M,\mathbb{C})) = 0$, then the curve $C$ is rational. If $\chi(C) < 0$, then $C$ is obtained from $\mathbb{C}$ by deleting at least two points, and $f^*H^1(C,\mathbb{C})$ is 0-isotropic with respect to $\cup_M$.*
(3) *Assume in addition that $\pi_1(M)$ is 1-formal. If the curve $C$ is compact of genus at least 1, then $f^*\colon H^2(C,\mathbb{C}) \to H^2(M,\mathbb{C})$ is injective, and so $f^*H^1(C,\mathbb{C})$ is 1-isotropic with respect to $\cup_M$.*

*Proof.* Recall that $f_\sharp\colon \pi_1(M) \to \pi_1(C)$ is surjective; hence $f^*\colon H^1(C,\mathbb{C}) \to H^1(M,\mathbb{C})$ is injective.

Part (1). Strictly speaking, a quasi-compact Kähler manifold $M$ does not have a *unique* mixed Hodge structure. Nevertheless, it inherits such a structure from each good compactification $\overline{M}$, by Deligne's construction in the smooth quasi-projective case, see [17].

By the admissibility condition on $f\colon M \to C$, there is a good compactification $\overline{M}$ such that $f$ extends to a regular morphism $\overline{f}\colon \overline{M} \to \overline{C}$. Fixing such an extension, the condition $W_1(H^1(M,\mathbb{C})) = H^1(M,\mathbb{C})$ simply means that $j^*(H^1(\overline{M},\mathbb{C})) = H^1(M,\mathbb{C})$, where $j\colon M \to \overline{M}$ is the inclusion. Since regular maps $f$ which extend to good compactifications of source and target obviously preserve weight filtrations, the mixed Hodge structure on $H^1(C,\mathbb{C})$ must be pure of weight 1, see [17]. If we write $C = \overline{C} \setminus A$, for some finite set $A$, then there is an exact Gysin sequence

$$0 \to H^1(\overline{C},\mathbb{C}) \to H^1(C,\mathbb{C}) \to H^0(A,\mathbb{C})(-1) \to H^2(\overline{C},\mathbb{C}) \to H^2(C,\mathbb{C}) \to 0\ ,$$

see for instance [19, p. 246]. But $H^0(A,\mathbb{C})(-1)$ is pure of weight 2, and so $H^1(C,\mathbb{C})$ is pure of weight 1 if and only if $|A| \leq 1$.

Part (2). By the same argument as before, we infer in this case that $H^1(C,\mathbb{C})$ should be pure of weight 2. The above Gysin sequence shows that $H^1(C,\mathbb{C})$ is pure of weight 2 if and only if $g(\overline{C})=0$, i.e., $\overline{C}=\mathbb{P}^1$. Finally, $\chi(C)<0$ implies $|A|\geq 3$.

Part (3). Set $G:=\pi_1(M)$, $\mathbb{T}_M:=\mathbb{T}_G$ and $\mathbb{T}:=\mathbb{T}_{\pi_1(C)}$. Note that $\dim\mathbb{T}>0$. Furthermore, the character torus $\mathbb{T}$ is embedded in $\mathbb{T}_M$, and its Lie algebra $T_1(\mathbb{T})$ is embedded in $T_1(\mathbb{T}_M)$, via the natural maps induced by $f$. By Theorem 5.2(2),

$$\dim H^1(M, {}_{f^*\rho}\mathbb{C}) = \dim H^1(C, {}_{\rho}\mathbb{C}),$$

for $\rho\in\mathbb{T}$ near 1 and different from 1, since both the surjectivity of $f_\sharp$ in Part (1), and the property from Part (2) do not require the assumption $\chi(C)<0$.

Applying Theorem 4.7 to both $G$ (using our 1-formality hypothesis), and $\pi_1(C)$ (using Example 2.5), we obtain from the above equality that

$$\dim H^1(H^\bullet(M,\mathbb{C}),\mu_{f^*z}) = \dim H^1(H^\bullet(C,\mathbb{C}),\mu_z),$$

for all $z\in H^1(C,\mathbb{C})$ near 0 and different from 0. Moreover, for any such $z$, a standard calculation shows $\dim H^1(H^\bullet(C,\mathbb{C}),\mu_z)=2g-2$, where $g=g(C)$.

Now suppose $f^*\colon H^2(C,\mathbb{C})\to H^2(M,\mathbb{C})$ were not injective. Then $f^*H^1(C,\mathbb{C})$ would be a 0-isotropic subspace of $H^1(M,\mathbb{C})$, containing $f^*(z)$. In turn, this would imply $\dim H^1(H^\bullet(M,\mathbb{C}),\mu_{f^*z})\geq 2g-1$, a contradiction. $\square$

We close this section by pointing out the subtlety of the injectivity property from Proposition 5.10(3).

**Example 5.11.** Let $L_g$ be the complex algebraic line bundle associated to the divisor given by a point on a projective smooth complex curve $C_g$ of genus $g\geq 1$. Denote by $M_g$ the total space of the $\mathbb{C}^*$-bundle associated to $L_g$. Clearly, $M_g$ is a smooth, quasi-projective manifold. (For $g=1$, this example was examined by Morgan in [55, p. 203].) Denote by $f_g\colon M_g\to C_g$ the natural projection. This map is a locally trivial fibration, which is admissible in the sense of Arapura [2]. Since the Chern class $c_1(L_g)\in H^2(C_g,\mathbb{Z})$ equals the fundamental class, it follows that $f_g^*\colon H^2(C_g,\mathbb{C})\to H^2(M_g,\mathbb{C})$ is the zero map. Set $G_g=\pi_1(M_g)$.

A straightforward analysis of the Serre spectral sequence associated to $f_g$, with arbitrary untwisted field coefficients, shows that $(f_g)_*\colon H_1(M_g,\mathbb{Z})\to H_1(C_g,\mathbb{Z})$ is an isomorphism, which identifies the respective character tori, to be denoted in the sequel by $\mathbb{T}_g$. This also implies that $W_1(H^1(M_g,\mathbb{C}))=H^1(M_g,\mathbb{C})$, since this property holds for the compact variety $C_g$.

We claim $f_g$ induces an isomorphism

$$H^1(C_g, {}_{\rho}\mathbb{C}) \xrightarrow{\simeq} H^1(M_g, {}_{f_g^*\rho}\mathbb{C}), \tag{5.2}$$

for all $\rho\in\mathbb{T}_g$. If $\rho=1$, this is clear. If $\rho\neq 1$, then $\mathrm{Hom}_{\mathbb{Z}\pi_1(C_g)}(\mathbb{Z},{}_{\rho}\mathbb{C})=0$, since the monodromy action of $\pi_1(C_g)$ on $\mathbb{Z}=H_1(\mathbb{C}^*,\mathbb{Z})$ is trivial. The claim follows from the 5-term exact sequence for twisted cohomology associated to the group extension $1\to\mathbb{Z}\to G_g\to\pi_1(C_g)\to 1$; see [36, VI.8(8.2)].

It follows that

$$\mathcal{V}_k(G_g) = \begin{cases} \mathbb{T}_g, & \text{for } 0 \le k \le 2g-2; \\ \{1\}, & \text{for } 2g-1 \le k \le 2g. \end{cases} \tag{5.3}$$

On the other hand, $\cup_{G_g} = 0$, since $f_g^* = 0$ on $H^2$. Therefore

$$\mathcal{R}_k(G_g) = \begin{cases} T_1(\mathbb{T}_g), & \text{for } 0 \le k \le 2g-1; \\ \{0\}, & \text{for } k = 2g. \end{cases} \tag{5.4}$$

By inspecting (5.3) and (5.4), we see that the tangent cone formula fails for $k = 2g-1$. Consequently, the (quasi-projective) group $G_g$ cannot be 1-formal. We thus see that the 1-formality hypothesis from Proposition 5.10(3) is essential for obtaining the injectivity property of $f^*$ on $H^2$.

**Remark 5.12.** It is easy to show that $f^*\colon H^2(C,\mathbb{C}) \to H^2(M,\mathbb{C})$ is injective when $M$ is compact. On the other hand, consider the following genus zero example, kindly provided to us by Morihiko Saito. Take $C = \mathbb{P}^1$ and $M = \mathbb{P}^1 \times \mathbb{P}^1 \setminus (C_1 \cup C_2)$, where $C_1 = \{\infty\} \times \mathbb{P}^1$ and $C_2$ is the diagonal in $\mathbb{P}^1 \times \mathbb{P}^1$. The projection of $M$ on the first factor has as image $\mathbb{C} = \mathbb{P}^1 \setminus \{\infty\}$ and affine lines as fibers; thus, $M$ is contractible. If we take $f\colon M \to \mathbb{P}^1$ to be the map induced by the second projection, we get an admissible map such that $f^*\colon H^2(C,\mathbb{C}) \to H^2(M,\mathbb{C})$ is not injective.

## 6. Position and resonance obstructions

In this section, we give our obstructions to realizing a finitely presented group $G$ as the fundamental group of a connected, quasi-compact Kähler manifold $M$. We start with a definition.

**Definition 6.1.** Let $\mu\colon H^1 \wedge H^1 \to H^2$ be a $\mathbb{C}$-linear map, with $\dim H^1 > 0$. Let $\mathcal{R}_k(\mu) \subset H^1$ be the corresponding resonance varieties, and let $\mathcal{R}_1(\mu) = \bigcup_\alpha \mathcal{R}^\alpha$ be the decomposition of $\mathcal{R}_1(\mu)$ into irreducible components. We say $\mu$ satisfies the *resonance obstructions* if the following conditions hold.

1. **Linearity:** Each component $\mathcal{R}^\alpha$ is a linear subspace of $H^1$.
2. **Isotropicity:** If $\mathcal{R}^\alpha \neq \{0\}$, then $\mathcal{R}^\alpha$ is a $p$-isotropic subspace of dimension at least $2p+2$, for some $p = p(\alpha) \in \{0,1\}$.
3. **Genericity:** If $\alpha \neq \beta$, then $\mathcal{R}^\alpha \bigcap \mathcal{R}^\beta = \{0\}$.
4. **Filtration by dimension:** For $1 \le k \le \dim H^1$,

$$\mathcal{R}_k(\mu) = \bigcup\nolimits_\alpha \mathcal{R}^\alpha,$$

   where the union is taken over all components $\mathcal{R}^\alpha$ such that $\dim \mathcal{R}^\alpha > k+p(\alpha)$. By convention, the union equals $\{0\}$ if the set of such components is empty.

Let $\mathcal{T} = \bigcup_\alpha \mathcal{T}^\alpha$ be the irreducible decomposition of a homogeneous variety $\mathcal{T} \subset H^1$. We say $\mathcal{T}$ satisfies the *position obstructions* with respect to $\mu$ if conditions **1**–**3** above hold, with $\mathcal{T}$ replacing $\mathcal{R}_1(\mu)$.

Note that the above resonance conditions depend only on the equivalence class of $\mu$, in the sense of Definition 5.8. Note also that $\mathcal{R}_1(\mu) = \emptyset$, if $H^1 = 0$, while $0 \in \mathcal{R}_1(\mu)$, if $\dim H^1 > 0$. If $\mu = \cup_G$, where $G = \pi_1(M)$, we will examine the above position conditions for $\mathcal{T} = TC_1(\mathcal{V}_1(G))$. Note that, again, $\mathcal{T} = \emptyset$, if $H^1 = 0$, while $0 \in \mathcal{T}$, if $H^1 \neq 0$.

The next three lemmas will be used in establishing the position obstruction from Theorem B(2).

**Lemma 6.2.** *Let $X$ be a connected quasi-compact Kähler manifold, $C$ a smooth curve and $f\colon X \to C$ a non-constant holomorphic mapping. Assume that $f$ admits a holomorphic extension $\hat{f}\colon \hat{X} \to \hat{C}$, where $\hat{X}$ (resp. $\hat{C}$) is a smooth compactification of $X$ (resp. $C$). Then the induced homomorphism in homology, $f_*\colon H_1(X, \mathbb{Z}) \to H_1(C, \mathbb{Z})$, has finite cokernel.*

*Proof.* Let $Y = \mathrm{Sing}(\hat{f})$ be the set of singular points of $\hat{f}$, i.e., the set of all points $x \in \hat{X}$ such that $d_x\hat{f} = 0$. Then $Y$ is a closed analytic subset of $\hat{X}$. Using Remmert's Theorem, we find that $Z = \hat{f}(Y)$ is a closed analytic subset of $\hat{C}$, and $\hat{f}(\hat{X}) = \hat{C}$. By Sard's Theorem, $Z \neq \hat{C}$, hence $Z$ is a finite set.

Let $B = (\hat{C} \setminus C) \cup Z$; set $C' = \hat{C} \setminus B$, and $\hat{X}' = \hat{X} \setminus \hat{f}^{-1}(B) = \hat{f}^{-1}(C')$. Then the restriction $\hat{f}'\colon \hat{X}' \to C'$ is a locally trivial fibration; its fiber is a compact manifold, and thus has only finitely many connected components. Using the tail end of the homotopy exact sequence of this fibration, we deduce that the induced homomorphism, $\hat{f}'_\sharp\colon \pi_1(\hat{X}') \to \pi_1(C')$, has image of finite index.

Now note that $i \circ \hat{f}' \circ k = f \circ j$, where $i\colon C' \to C$, $j\colon X \setminus \hat{f}^{-1}(B) \to X$, and $k\colon X \setminus \hat{f}^{-1}(B) \to \hat{X}'$ are the inclusion maps. From the above, it follows that $f_*\colon H_1(X, \mathbb{Z}) \to H_1(C, \mathbb{Z})$ has image of finite index. $\square$

Let $\mathcal{V}^\alpha$ and $\mathcal{V}^\beta$ be two distinct, positive-dimensional, irreducible components of $\mathcal{V}_1(\pi_1(M))$ containing 1. Realize them by pull-back, via admissible maps, $f_\alpha\colon M \to C_\alpha$ and $f_\beta\colon M \to C_\beta$, as in Theorem 5.2(1). We know that generically (that is, for $t \in C_\alpha \setminus B_\alpha$, where $B_\alpha$ is finite) the fiber $f_\alpha^{-1}(t)$ is smooth and irreducible.

**Lemma 6.3.** *In the above setting, there exists $t \in C_\alpha \setminus B_\alpha$ such that the restriction of $f_\beta$ to $f_\alpha^{-1}(t)$ is non-constant.*

*Proof.* Assume $f_\beta$ has constant value, $h(t)$, on the fiber $f_\alpha^{-1}(t)$, for $t \in C_\alpha \setminus B_\alpha$. We first claim that this implies the existence of a continuous extension, $h\colon C_\alpha \to C_\beta$, with the property that $h \circ f_\alpha = f_\beta$.

Indeed, let us pick an arbitrary special value, $t_0 \in B_\alpha$, together with a sequence of generic values, $t_n \in C_\alpha \setminus B_\alpha$, converging to $t_0$. For any $x \in f_\alpha^{-1}(t_0)$, note that the

order at $x$ of the holomorphic function $f_\alpha$ is finite. Hence, we may find a sequence, $x_n \to x$, such that $f_\alpha(x_n) = t_n$. By our assumption, $f_\beta(x) = \lim h(t_n)$, independently of $x$, which proves the claim.

At the level of character tori, the fact that $h \circ f_\alpha = f_\beta$ implies $\mathcal{V}^\beta = f_\beta^* \mathbb{T}_{\pi_1(C_\beta)} \subset f_\alpha^* \mathbb{T}_{\pi_1(C_\alpha)} = \mathcal{V}^\alpha$, a contradiction. □

**Lemma 6.4.** *Let $\mathcal{V}^\alpha$ and $\mathcal{V}^\beta$ be two distinct irreducible components of $\mathcal{V}_1(\pi_1(M))$ containing 1. Then $\mathcal{V}^\alpha \cap \mathcal{V}^\beta$ is finite.*

*Proof.* We may suppose that both components are positive-dimensional. Lemma 6.3 guarantees the existence of a generic fiber of $f_\alpha$, say $F_\alpha$, with the property that the restriction of $f_\beta$ to $F_\alpha$, call it $g\colon F_\alpha \to C_\beta$, is non-constant. By Lemma 6.2, there exists a positive integer $m$ with the property that

$$m \cdot H_1(C_\beta, \mathbb{Z}) \subset \operatorname{im}(g_*)\,. \tag{6.1}$$

We will finish the proof by showing that $\rho^m = 1$, for any $\rho \in \mathcal{V}^\alpha \cap \mathcal{V}^\beta$.

To this end, write $\rho = \rho_\beta \circ (f_\beta)_*$, with $\rho_\beta \in \mathbb{T}_{\pi_1(C_\beta)}$. For an arbitrary element $a \in H_1(M, \mathbb{Z})$, we have $\rho^m(a) = \rho_\beta(m \cdot (f_\beta)_* a)$. From (6.1), it follows that $m \cdot (f_\beta)_* a = (f_\beta)_*(j_\alpha)_* b$, for some $b \in H_1(F_\alpha, \mathbb{Z})$, where $j_\alpha\colon F_\alpha \hookrightarrow M$ is the inclusion. On the other hand, we may also write $\rho = \rho_\alpha \circ (f_\alpha)_*$, with $\rho_\alpha \in \mathbb{T}_{\pi_1(C_\alpha)}$. Hence, $\rho^m(a) = \rho_\alpha((f_\alpha)_*(j_\alpha)_* b) = \rho_\alpha(0) = 1$, as claimed. □

The next result establishes Parts (1)–(4) of Theorem B from the Introduction.

**Theorem 6.5.** *Let $M$ be a connected, quasi-compact Kähler manifold. Set $G = \pi_1(M)$ and assume $b_1(G) > 0$. Let $\{\mathcal{V}^\alpha\}$ be the irreducible components of $\mathcal{V}_1(G)$ containing 1. Then the following hold.*

1. *The tangent spaces $\mathcal{T}^\alpha := T_1(\mathcal{V}^\alpha)$ are the irreducible components of $\mathcal{T} := TC_1(\mathcal{V}_1(G))$.*
2. *If $G$ is 1-formal, $\{\mathcal{T}^\alpha\}$ is the family of irreducible components of $\mathcal{R}_1(G)$.*
3. *The variety $\mathcal{T}$ satisfies the position obstructions* **1**–**3** *from Definition 6.1, with respect to $\cup_G\colon H^1(G,\mathbb{C}) \wedge H^1(G,\mathbb{C}) \to H^2(G,\mathbb{C})$.*
4. *If $G$ is 1-formal, $\cup_G$ verifies the resonance conditions* **1**–**4** *from 6.1.*

*Proof.* Part (1). As noticed before, $\mathcal{T} = \bigcup_\alpha \mathcal{T}^\alpha$. By [37, 13.1], $\mathcal{T} = \bigcup_\alpha \mathcal{T}^\alpha$ is the decomposition of $\mathcal{T}$ into irreducible components.

Part (2). If $\mathcal{R}_1(G) = 0$, then $\mathcal{T} = 0$, and there is nothing to prove. If $\mathcal{R}_1(G) \neq 0$, the statement follows from Corollary 5.3.

Part (3). Property **1** is clear. As for property **2**, we have $\mathcal{V}^\alpha = f_\alpha^* \mathbb{T}_{\pi_1(C_\alpha)}$, where $f_\alpha$ is admissible and $\chi(C_\alpha) < 0$; see Theorem 5.2(1). Therefore, $\mathcal{T}^\alpha = f_\alpha^* H^1(C_\alpha, \mathbb{C})$. If the curve $C_\alpha$ is non-compact, the subspace $\mathcal{T}^\alpha$ is clearly isotropic, and $\dim \mathcal{T}^\alpha = b_1(C_\alpha) \geq 2$. If $C_\alpha$ is compact and $f_\alpha^*$ is zero on $H^2(C_\alpha, \mathbb{C})$, we obtain the same conclusion as before. Finally, if $C_\alpha$ is compact and $f_\alpha^*$ is non-zero on $H^2(C_\alpha, \mathbb{C})$,

then plainly $\mathcal{T}^\alpha$ is 1-isotropic and $\dim \mathcal{T}^\alpha = b_1(C_\alpha) \geq 4$. The isotropicity property is thus established.

By [37, Theorem 12.5], property **3** is equivalent to the fact that $\mathcal{V}^\alpha \cap \mathcal{V}^\beta$ is finite. Thus, the genericity property follows from Lemma 6.4.

Part (4). By Parts (1)–(3), the resonance conditions **1**–**3** are verified by $\cup_G$. Property **4** may be checked as follows. We may assume $\mathcal{R}_1(G) \neq \{0\}$, since otherwise the property holds trivially. We may also assume $k < b_1(G)$, since otherwise $\mathcal{R}_k(G) = \{0\}$, and there is nothing to prove.

To prove the desired equality, we have to check first that any non-zero element $u \in \mathcal{R}_k(G)$ belongs to some $\mathcal{R}^\alpha$ with $\dim \mathcal{R}^\alpha > k$. Definition (4.2) guarantees the existence of elements $v_1, \dots, v_k \in H^1(G, \mathbb{C})$ with $v_i \cup u = 0$, and such that $u, v_1, \dots, v_k$ are linearly independent. Since the subspaces $\langle u, v_i \rangle$ spanned by the pairs $\{u, v_i\}$ are clearly contained in $\mathcal{R}_1(G)$, it follows that $\langle u, v_i \rangle \subset \mathcal{R}^{\alpha_i}$. Necessarily $\alpha_1 = \cdots = \alpha_k := \alpha$, since otherwise property **3** would be violated. This proves $u \in \mathcal{R}^\alpha$, with $\dim \mathcal{R}^\alpha > k$.

Now, if $p(\alpha) = 1$, then $\dim \mathcal{R}^\alpha > k + 1$. For otherwise, we would have $\mathcal{R}^\alpha = u^\perp_{\mathcal{R}^\alpha}$, which would violate the non-degeneracy property from Definition 5.6(ii). Finally, that $\dim \mathcal{R}^\alpha > k + p(\alpha)$ implies $\mathcal{R}^\alpha \subset \mathcal{R}_k(G)$ follows at once from Lemma 5.5. □

We now relate the dimensions of the cohomology groups $H^1(H^\bullet(M, \mathbb{C}), \mu_z)$ and $H^1(M, {}_\rho\mathbb{C})$ corresponding to $z \in \operatorname{Hom}(G, \mathbb{C}) \backslash \{0\}$ and $\rho = \exp(z) \in \operatorname{Hom}(G, \mathbb{C}^*) \backslash \{1\}$, to the dimension and isotropicity of the resonance component $\mathcal{R}^\alpha$ to which $z$ belongs.

**Proposition 6.6.** *Let $M$ be a connected quasi-compact Kähler manifold, with fundamental group $G$, and first resonance variety $\mathcal{R}_1(G) = \bigcup_\alpha \mathcal{R}^\alpha$. If $G$ is 1-formal, then the following hold.*

1. *If $z \in \mathcal{R}^\alpha$ and $z \neq 0$, then $\dim H^1(H^\bullet(M, \mathbb{C}), \mu_z) = \dim \mathcal{R}^\alpha - p(\alpha) - 1$.*
2. *If $\rho \in \exp(\mathcal{R}^\alpha)$ and $\rho \neq 1$, then $\dim H^1(M, {}_\rho\mathbb{C}) \geq \dim \mathcal{R}^\alpha - p(\alpha) - 1$, with equality for all except finitely many local systems $\rho$.*

*Proof.* Part (1). Recall that $\mathcal{R}^\alpha = f_\alpha^* H^1(C_\alpha, \mathbb{C})$. Exactly as in the proof of Proposition 5.10(3), we infer that

$$\dim H^1(M, {}_\rho\mathbb{C}) = \dim H^1(H^\bullet(M, \mathbb{C}), \mu_z) = \dim H^1(H^\bullet(C_\alpha, \mathbb{C}), \mu_\zeta), \tag{6.2}$$

where $z = f_\alpha^* \zeta$ and $\rho = \exp(z)$, for all $z \in \mathcal{R}^\alpha$ near 0 and different from 0. Clearly

$$\dim H^1(H^\bullet(C_\alpha, \mathbb{C}), \mu_\zeta) = \dim \mathcal{R}^\alpha - p(\alpha) - 1, \tag{6.3}$$

if $\zeta \neq 0$. Since plainly $\dim H^1(H^\bullet(M, \mathbb{C}), \mu_z) = \dim H^1(H^\bullet(M, \mathbb{C}), \mu_{\lambda z})$, for all $\lambda \in \mathbb{C}^*$, equations (6.2) and (6.3) finish the proof of (1).

Part (2). Starting from the standard presentation of the group $\pi_1(C_\alpha)$, a Fox calculus computation shows that $\dim H^1(C_\alpha, {}_{\rho'}\mathbb{C}) = \dim \mathcal{R}^\alpha - p(\alpha) - 1$, provided $\rho' \neq 1$. By Theorem 5.2(2), the equality $\dim H^1(M, {}_\rho\mathbb{C}) = \dim \mathcal{R}^\alpha - p(\alpha) - 1$ holds for all but finitely many local systems $\rho \in \exp(\mathcal{R}^\alpha)$. By semi-continuity (see Remark 4.4), the inequality $\dim H^1(M, {}_\rho\mathbb{C}) \geq \dim \mathcal{R}^\alpha - p(\alpha) - 1$ holds for all $\rho \in \exp(\mathcal{R}^\alpha)$. □

Using Proposition 5.10, we obtain the following corollary to Theorem 6.5.

**Corollary 6.7.** *Let $M$ be a connected quasi-compact Kähler manifold, with fundamental group $G$, and first resonance variety $\mathcal{R}_1(G) = \bigcup_\alpha \mathcal{R}^\alpha$. Assume $b_1(G) > 0$ and $\mathcal{R}_1(G) \neq \{0\}$.*

(1) *If $M$ is compact then $G$ is 1-formal, and $\cup_G$ satisfies the resonance obstructions. Moreover, each component $\mathcal{R}^\alpha$ is 1-isotropic, with $\dim \mathcal{R}^\alpha = 2g_\alpha \geq 4$.*
(2) *If $W_1(H^1(M,\mathbb{C})) = 0$ then $G$ is 1-formal, and $\cup_G$ satisfies the resonance obstructions. Moreover, each component $\mathcal{R}^\alpha$ is 0-isotropic, with $\dim \mathcal{R}^\alpha \geq 2$.*
(3) *If $W_1(H^1(M,\mathbb{C})) = H^1(M,\mathbb{C})$ and $G$ is 1-formal, then $\dim \mathcal{R}^\alpha = 2g_\alpha \geq 2$, for all $\alpha$.*

Next, we sharpen a result of Arapura [2, Corollary V.1.9], under a 1-formality assumption, thereby extending known characterizations of 'fibered' Kähler groups (see [1, §2.3] for a survey).

**Corollary 6.8.** *Let $M$ be a connected quasi-compact Kähler manifold. Suppose the group $G = \pi_1(M)$ is 1-formal. The following are then equivalent.*

(i) *There is an admissible map, $f\colon M \to C$, onto a smooth complex curve $C$ with $\chi(C) < 0$.*
(ii) *There is an epimorphism, $\varphi\colon G \twoheadrightarrow \pi_1(C)$, onto the fundamental group of a smooth complex curve $C$ with $\chi(C) < 0$.*
(iii) *There is an epimorphism, $\varphi\colon G \twoheadrightarrow \mathbb{F}_r$, onto a free group of rank $r \geq 2$.*
(iv) *The resonance variety $\mathcal{R}_1(G)$ strictly contains $\{0\}$.*

*Proof.* The equivalence of the first three properties is proved in [2, Corollary V.1.9], without our additional 1-formality hypothesis. The implication (iii) $\Rightarrow$ (iv) also holds in general; this may be seen by using the $\cup_G$-isotropic subspace $\varphi^* H^1(\mathbb{F}_r,\mathbb{C}) \subset H^1(G,\mathbb{C})$. Finally, assuming $G$ is 1-formal, the implication (iv) $\Rightarrow$ (i) follows from Corollary 5.3. □

The equivalence (iii) $\Leftrightarrow$ (iv) above finishes the proof of Theorem B from the Introduction.

We close this section with a pair of examples showing that both the quasi-Kähler and the 1-formality assumptions are needed in order for this equivalence to hold.

**Example 6.9.** Consider the smooth, quasi-projective variety $M_1$ from Example 5.11 (the complex version of the Heisenberg manifold). As mentioned previously, the group $G_1 = \pi_1(M_1)$ is not 1-formal. On the other hand, it is well-known that $G_1$ is a nilpotent group. Therefore, property (iii) fails for $G_1$. Nevertheless, $\mathcal{R}_1(G_1) = \mathbb{C}^2$, by virtue of (5.4), and so property (iv) holds for $G_1$.

**Example 6.10.** Let $N_h$ be the non-orientable surface of genus $h \geq 1$, that is, the connected sum of $h$ real projective planes. It is readily seen that $N_h$ has the rational homotopy type of a wedge of $h-1$ circles. Hence $N_h$ is a formal space, and so $\pi_1(N_h)$

is a 1-formal group. Moreover, $\mathcal{R}_1(\pi_1(N_h)) = \mathcal{R}_1(\mathbb{F}_{h-1})$, and so $\mathcal{R}_1(\pi_1(N_h)) = \mathbb{C}^{h-1}$, provided $h \geq 3$. Thus, property (iv) holds for all groups $\pi_1(N_h)$ with $h \geq 3$.

Now suppose there is an epimorphism $\varphi\colon \pi_1(N_h) \twoheadrightarrow \mathbb{F}_r$ with $r \geq 2$, as in (iii). Then the subspace $\varphi^* H^1(\mathbb{F}_r, \mathbb{Z}_2) \subset H^1(\pi_1(N_h), \mathbb{Z}_2)$ has dimension at least 2, and is isotropic with respect to $\cup_{\pi_1(N_h)}$. Hence, $h \geq 4$, by Poincaré duality with $\mathbb{Z}_2$ coefficients.

Focussing on the case $h = 3$, we see that the group $\pi_1(N_3)$ is 1-formal, yet the implication (iv) $\Rightarrow$ (iii) from Corollary 6.8 fails for this group. It follows that $\pi_1(N_3)$ cannot be realized as the fundamental group of a quasi-compact Kähler manifold. Note that this assertion is *not* a consequence of Theorem 6.5(4); indeed, $\cup_{\pi_1(N_h)} \simeq \cup_{\mathbb{F}_{h-1}}$ (over $\mathbb{C}$), while $\mathbb{F}_{h-1} = \pi_1(\mathbb{P}^1 \setminus \{h \text{ points}\})$, for all $h \geq 1$.

## 7. Wedges and products

In this section, we analyze products and coproducts of groups, together with their counterparts at the level of first resonance varieties. Using our obstructions, we obtain conditions for realizability of free products of groups by quasi-compact Kähler manifolds.

7.1. **Products, coproducts, and 1-formality.** Let $\mathbb{F}(X)$ be the free group on a finite set $X$, and let $\mathbb{L}^*(X)$ be the free Lie algebra on $X$, over a field $\Bbbk$ of characteristic 0. Denote by $\widehat{\mathbb{L}}(X)$ the Malcev Lie algebra obtained from $\mathbb{L}^*(X)$ by completion with respect to the degree filtration. Define the group homomorphism $\kappa_X\colon \mathbb{F}(X) \to \mathsf{exp}(\widehat{\mathbb{L}}(X))$ by $\kappa_X(x) = x$ for $x \in X$. Standard commutator calculus [41] shows that

$$\mathrm{gr}^*(\kappa_X)\colon\ \mathrm{gr}^*(\mathbb{F}(X)) \otimes \Bbbk \xrightarrow{\simeq} \mathrm{gr}^*_F(\widehat{\mathbb{L}}(X)) \tag{7.1}$$

is an isomorphism. It follows from [63, Appendix A] that $\kappa_X$ is a Malcev completion.

Now let $G$ be a finitely presented group, with presentation $G = \langle x_1, \dots, x_s \mid w_1, \dots, w_r\rangle$, or, for short, $G = \mathbb{F}(X)/\langle \mathbf{w}\rangle$. Denote by $\langle\!\langle \mathbf{w}\rangle\!\rangle$ the closed Lie ideal of $\widehat{\mathbb{L}}(X)$ generated by $\kappa_X(w_1), \dots, \kappa_X(w_r)$, and consider the group morphism induced by $\kappa_X$,

$$\kappa_G\colon G \to \mathsf{exp}\big(\widehat{\mathbb{L}}(X)/\langle\!\langle \mathbf{w}\rangle\!\rangle\big). \tag{7.2}$$

It follows from [57] that $\kappa_G$ is a Malcev completion for $G$. (For the purposes of that paper, it was assumed that $G_{\mathrm{ab}}$ had no torsion, see [57, Example 2.1]. Actually, the proof of the Malcev completion property applies verbatim to the general case, see [57, Theorem 2.2].)

**Proposition 7.2.** *If $G_1$ and $G_2$ are 1-formal groups, then their coproduct $G_1 * G_2$ and their product $G_1 \times G_2$ are again 1-formal groups.*

*Proof.* First consider two arbitrary finitely presented groups, with presentations $G_1 = \mathbb{F}(X)/\langle \mathbf{u}\rangle$ and $G_2 = \mathbb{F}(Y)/\langle \mathbf{v}\rangle$. Then $G_1 * G_2 = \mathbb{F}(X \cup Y)/\langle \mathbf{u}, \mathbf{v}\rangle$. It follows from (7.2) that $E_{G_1 * G_2} = E_{G_1} \coprod E_{G_2}$, the coproduct Malcev Lie algebra.

On the other hand, $G_1 \times G_2 = \mathbb{F}(X \cup Y)/\langle \mathbf{u}, \mathbf{v}, (x,y); x \in X, y \in Y\rangle$, and so, by the same reasoning, $E_{G_1\times G_2} = \widehat{\mathbb{L}}(X \cup Y)/\langle\langle \kappa_X(\mathbf{u}), \kappa_Y(\mathbf{v}), (x,y); x \in X, y \in Y\rangle\rangle$. Using the Campbell-Hausdorff formula, we may replace each CH-group commutator $(x, y)$ with the corresponding Lie bracket, $[x, y]$; see [58, Lemma 2.5] for details. We conclude that $E_{G_1\times G_2} = E_{G_1} \prod E_{G_2}$, the product Malcev Lie algebra.

Now assume $G_1$ and $G_2$ are 1-formal. In view of Lemma 2.9, we may write $E_{G_1} = \widehat{\mathbb{L}}(X')/\langle\langle \mathbf{u}'\rangle\rangle$ and $E_{G_2} = \widehat{\mathbb{L}}(Y')/\langle\langle \mathbf{v}'\rangle\rangle$, where the defining relations $\mathbf{u}'$ and $\mathbf{v}'$ are quadratic. Hence

$$E_{G_1 * G_2} = \widehat{\mathbb{L}}(X' \cup Y')/\langle\langle \mathbf{u}', \mathbf{v}'\rangle\rangle,$$

$$E_{G_1\times G_2} = \widehat{\mathbb{L}}(X' \cup Y')/\langle\langle \mathbf{u}', \mathbf{v}', [x', y']; x' \in X', y' \in Y'\rangle\rangle.$$

Since the relations in these presentations are clearly quadratic, the 1-formality of both $G_1 * G_2$ and $G_1 \times G_2$ follows from Lemma 2.9. $\square$

7.3. **Products, coproducts, and resonance.** Let $U^i$, $V^i$ $(i = 1, 2)$ be complex vector spaces. Given two $\mathbb{C}$-linear maps, $\mu_U : U^1 \wedge U^1 \to U^2$ and $\mu_V : V^1 \wedge V^1 \to V^2$, set $W^i = U^i \oplus V^i$, and define $\mu_U * \mu_V : W^1 \wedge W^1 \to W^2$ as follows:

$$\mu_U * \mu_V|_{U^1\wedge U^1} = \mu_U, \quad \mu_U * \mu_V|_{V^1\wedge V^1} = \mu_V, \quad \mu_U * \mu_V|_{U^1\wedge V^1} = 0.$$

When $\mu_U = \cup_{G_1}$ and $\mu_V = \cup_{G_2}$, then clearly $\mu_U * \mu_V = \cup_{G_1*G_2}$, since $K(G_1 * G_2, 1) = K(G_1, 1) \vee K(G_2, 1)$.

**Lemma 7.4.** *Suppose $U^1 \neq 0$, $V^1 \neq 0$, and $\mu_U * \mu_V$ satisfies the second resonance obstruction from Definition 6.1. Then $\mu_U = \mu_V = 0$.*

*Proof.* Set $\mu := \mu_U * \mu_V$. We know $\mathcal{R}_1(\mu) = W^1$, by [61, Lemma 5.2]. If $\mu \neq 0$, then $\mu$ is 1-isotropic, with 1-dimensional image. It follows that either $\mu_U = 0$ or $\mu_V = 0$. In either case, $\mu$ fails to be non-degenerate, a contradiction. Thus, $\mu = 0$, and so $\mu_U = \mu_V = 0$. $\square$

Next, given $\mu_U$ and $\mu_V$ as above, set $Z^1 = U^1 \oplus V^1$ and $Z^2 = U^2 \oplus V^2 \oplus (U^1 \otimes V^1)$, and define $\mu_U \times \mu_V : Z^1 \wedge Z^1 \to Z^2$ as follows. As before, the restrictions of $\mu_U \times \mu_V$ to $U^1 \wedge U^1$ and $V^1 \wedge V^1$ are given by $\mu_U$ and $\mu_V$, respectively. On the other hand, $\mu_U \times \mu_V (u \wedge v) = u \otimes v$, for $u \in U^1$ and $v \in V^1$. Finally, if $\mu_U = \cup_{G_1}$ and $\mu_V = \cup_{G_2}$, then $\mu_U \times \mu_V = \cup_{G_1\times G_2}$, since $K(G_1 \times G_2, 1) = K(G_1, 1) \times K(G_2, 1)$.

**Lemma 7.5.** *With notation as above, $\mathcal{R}_1(\mu_U \times \mu_V) = \mathcal{R}_1(\mu_U) \times \{0\} \cup \{0\} \times \mathcal{R}_1(\mu_V)$.*

*Proof.* Set $\mu = \mu_U \times \mu_V$. The inclusion $\mathcal{R}_1(\mu) \supset \mathcal{R}_1(\mu_U) \times \{0\} \cup \{0\} \times \mathcal{R}_1(\mu_V)$ is obvious. To prove the other inclusion, assume $\mathcal{R}_1(\mu) \neq 0$ (otherwise, there is nothing to prove), and pick $0 \neq a + b \in \mathcal{R}_1(\mu)$, with $a \in U^1$ and $b \in V^1$. By definition of $\mathcal{R}_1(\mu)$, there is $x + y \in U^1 \oplus V^1$ such that $(a + b) \wedge (x + y) \neq 0$ and

$$\mu((a + b) \wedge (x + y)) = \mu_U(a \wedge x) + \mu_V(b \wedge y) + a \otimes y - x \otimes b = 0. \tag{7.3}$$

In particular, $a \otimes y = x \otimes b$. There are several cases to consider.

If $a \neq 0$ and $b \neq 0$, we must have $x = \lambda a$ and $y = \lambda b$, for some $\lambda \in \mathbb{C}$, and so $(a+b) \wedge (x+y) = (a+b) \wedge \lambda(a+b) = 0$, a contradiction.

If $b = 0$, then $a \neq 0$ and (7.3) forces $y = 0$ and $\mu_U(a \wedge x) = 0$. Since $(a+b) \wedge (x+y) = a \wedge x \neq 0$, it follows that $a \in \mathcal{R}_1(\mu_U)$, as needed. The other case, $a = 0$, leads by the same reasoning to $b \in \mathcal{R}_1(\mu_V)$. □

If $G_1$ and $G_2$ are finitely presented groups, Lemma 7.5 implies that $\mathcal{R}_1(G_1 \times G_2) = \mathcal{R}_1(G_1) \times \{0\} \cup \{0\} \times \mathcal{R}_1(G_2)$. An analogous formula holds for the characteristic varieties: $\mathcal{V}_1(G_1 \times G_2) = \mathcal{V}_1(G_1) \times \{1\} \cup \{1\} \times \mathcal{V}_1(G_2)$, see [14, Theorem 3.2].

7.6. **Quasi-projectivity of coproducts.** Here is an application of Theorem B. It is inspired by a result of M. Gromov, who proved in [32] that no non-trivial free product of groups can be realized as the fundamental group of a compact Kähler manifold. We need two lemmas.

**Lemma 7.7.** *Let $G$ be a finitely presented, commutator relators group (that is, $G = \mathbb{F}(X)/\langle \mathbf{w} \rangle$, with $X$ and $\mathbf{w}$ finite, and $\mathbf{w} \subset \Gamma_2\mathbb{F}(X)$). Suppose $G$ is 1-formal, and $\cup_G = 0$. Then $G$ is a free group.*

*Proof.* Pick a presentation $G = \mathbb{F}(X)/\langle \mathbf{w} \rangle$, with all relators $w_i$ words in the commutators $(g,h)$, where $g, h \in \mathbb{F}(X)$. We have $E_G = \widehat{\mathfrak{H}}(G)$, by the 1-formality of $G$, and $\mathfrak{H}(G) = \mathbb{L}(X)$, by the vanishing of $\cup_G$. Hence, $E_G = \widehat{\mathbb{L}}(X)$. On the other hand, (7.2) implies $E_G = \widehat{\mathbb{L}}(X)/\langle\langle \mathbf{w} \rangle\rangle$. We thus obtain a filtered Lie algebra isomorphism, $\widehat{\mathbb{L}}(X) \xrightarrow{\simeq} \widehat{\mathbb{L}}(X)/\langle\langle \mathbf{w} \rangle\rangle$.

Taking quotients relative to the respective Malcev filtrations and comparing vector space dimensions, we see that $\kappa_X(w_i) \in \bigcap_{k\geq 1} F_k\widehat{\mathbb{L}}(X) = 0$, for all $i$. A well-known result of Magnus (see [47]) says that $\operatorname{gr}^*(\mathbb{F}(X))$ is a torsion-free graded abelian group. We infer from (7.1) that $w_i \in \bigcap_{k\geq 1} \Gamma_k\mathbb{F}(X)$, for all $i$. Another well-known result of Magnus (see [47]) insures that $\mathbb{F}(X)$ is residually nilpotent, i.e., $\bigcap_{k\geq 1} \Gamma_k\mathbb{F}(X) = 1$. Hence, $w_i = 1$, for all $i$, and so $G = \mathbb{F}(X)$. □

**Lemma 7.8.** *Let $G_1$ and $G_2$ be finitely presented groups with non-zero first Betti number. Then $\mathcal{V}_1(G_1 * G_2) = \mathbb{T}_{G_1 * G_2}$.*

*Proof.* Let $G = \langle x_1, \dots, x_s \mid w_1, \dots, w_r \rangle$ be an arbitrary finitely presented group, and let $\rho \in \mathbb{T}_G$ be an arbitrary character. Recall from Remarks 4.4 and 4.14 that $\rho \in \mathcal{V}_1(G)$ if and only if $b_1(G,\rho) > 0$, where $b_1(G,\rho) := \dim\ker d_1(\rho) - \operatorname{rank} d_2(\rho)$. Moreover, the linear map $d_1(\rho)\colon \mathbb{C}^s \to \mathbb{C}$ sends the basis element corresponding to the generator $x_i$ to $\rho(x_i) - 1$, while the linear map $d_2(\rho)\colon \mathbb{C}^r \to \mathbb{C}^s$ is given by the evaluation at $\rho$ of the matrix of free derivatives of the relators, $\left(\frac{\partial w_j}{\partial x_i}(\rho)\right)$; see Fox [30].

For $G = G_1 * G_2$, write $\rho = (\rho_1, \rho_2)$, with $\rho_i \in \mathbb{T}_{G_i}$. We then have $d_j(\rho) = d_j(\rho_1) + d_j(\rho_2)$, for $j = 1, 2$. Hence, $b_1(G,\rho) = b_1(G_1,\rho_1) + b_1(G_2,\rho_2) + 1$, if both $\rho_1$ and $\rho_2$ are different from 1, and otherwise $b_1(G,\rho) = b_1(G_1,\rho_1) + b_1(G_2,\rho_2)$. Since $b_1(G_i, 1) = b_1(G_i) > 0$, the claim follows. □

**Theorem 7.9.** *Let $G_1$ and $G_2$ be finitely presented groups with non-zero first Betti number.*

(1) *If the coproduct $G_1 * G_2$ is quasi-Kähler, then $\cup_{G_1} = \cup_{G_2} = 0$.*
(2) *Assume moreover that $G_1$ and $G_2$ are 1-formal, presented by commutator relators only. Then $G_1 * G_2$ is a quasi-Kähler group if and only if both $G_1$ and $G_2$ are free.*

*Proof.* Part (1). Set $G = G_1 * G_2$. From Lemma 7.8, we know that there is just one irreducible component of $\mathcal{V}_1(G)$ containing 1, namely $\mathcal{V} = \mathbb{T}^0_G$, the component of the identity in the character torus. Hence, $T_1(\mathcal{V}) = H^1(G, \mathbb{C})$. Libgober's result from [44] implies then that $\mathcal{R}_1(G) = H^1(G, \mathbb{C})$. If $G$ is quasi-Kähler, Theorem B(1) may be invoked to infer that $\cup_G$ satisfies the isotropicity resonance obstruction from Definition 6.1. The conclusion follows from Lemma 7.4.

Part (2). If $G_1$ and $G_2$ are free, then $G_1 * G_2$ is also free (of finite rank), thus quasi-projective. For the converse, use Part (1) to deduce that $\cup_{G_1} = \cup_{G_2} = 0$, and then apply Lemma 7.7. □

Let $\mathcal{C}$ be the class of fundamental groups of complex projective curves of non-zero genus. Each $G \in \mathcal{C}$ is a 1-formal group, admitting a presentation with a single commutator relator, and is not free (for instance, since $\cup_G \neq 0$). Proposition 7.2 and Theorem 7.9 yield the following corollary.

**Corollary 7.10.** *If $G_1, G_2 \in \mathcal{C}$, then $G_1 * G_2$ is a 1-formal group, yet $G_1 * G_2$ is not realizable as the fundamental group of a smooth, quasi-projective variety $M$.*

This shows that 1-formality and quasi-projectivity may exhibit contrasting behavior with respect to the coproduct operation for groups.

## 8. Arrangements of real planes

Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an arrangement of planes in $\mathbb{R}^4$, meeting transversely at the origin. By intersecting $\mathcal{A}$ with a 3-sphere about 0, we obtain a link $L$ of $n$ great circles in $S^3$. It is readily seen that the complement $M$ of the arrangement deform-retracts onto the complement of the link. Moreover, the fundamental group $G = \pi_1(M)$ has the structure of a semidirect product of free groups, $G = F_{n-1} \rtimes \mathbb{Z}$, and $M$ is a $K(G, 1)$. For details, see [75], [50].

**Example 8.1.** Let $\mathcal{A} = \mathcal{A}(2134)$ be the arrangement defined in complex coordinates on $\mathbb{R}^4 = \mathbb{C}^2$ by the half-holomorphic function $Q(z, w) = zw(z - w)(z - 2\bar{w})$; see Ziegler [75, Example 2.2]. Using a computation from [50, Example 5.10], we obtain the following presentation for the fundamental group of the complement

$$G = \langle x_1, x_2, x_3, x_4 \mid (x_1, x_3^2 x_4),\ (x_2, x_4),\ (x_3, x_4) \rangle.$$

It can be seen that $E_G = \widehat{\mathbb{L}}(x_1, x_2, x_3, x_4)/\langle\langle 2[x_1, x_3] + [x_1, x_4], [x_2, x_4], [x_3, x_4] \rangle\rangle$; thus, $G$ is 1-formal. The resonance variety $\mathcal{R}_1(G) \subset \mathbb{C}^4$ has two components, $\mathcal{R}^\alpha = \{x \mid$

$x_4 = 0\}$ and $\mathcal{R}^\beta = \{x \mid x_4 + 2x_3 = 0\}$, and the tangent cone formula holds for $G$. Though both components of $\mathcal{R}_1(G)$ are linear, the other three resonance obstructions are violated:

- The subspaces $\mathcal{R}^\alpha$ and $\mathcal{R}^\beta$ are neither 0-isotropic, nor 1-isotropic.
- $\mathcal{R}^\alpha \cap \mathcal{R}^\beta = \{x \mid x_3 = x_4 = 0\}$, which is not equal to $\{0\}$.
- $\mathcal{R}_2(G) = \{x \mid x_1 = x_3 = x_4 = 0\} \cup \{x \mid x_2 = x_3 = x_4 = 0\}$, and neither of these components equals $\mathcal{R}^\alpha$ or $\mathcal{R}^\beta$.

Thus, $G$ is not the fundamental group of any smooth quasi-projective variety.

Let $\mathcal{A}$ be an arrangement of transverse planes in $\mathbb{R}^4$, with complement $M$. From the point of view of two classical invariants—the associated graded Lie algebra, and the Chen Lie algebra—the group $G = \pi_1(M)$ behaves like a 1-formal group. Indeed, the associated link $L$ has all linking numbers equal to $\pm 1$, in particular, the linking graph of $L$ is connected. Thus, $\mathrm{gr}^*(G) \otimes \mathbb{Q} \cong \mathfrak{H}_G$ and $\mathrm{gr}^*(G/G'') \otimes \mathbb{Q} \cong \mathfrak{H}_G/\mathfrak{H}''_G$, as graded Lie algebras, by [48, Corollary 6.2] and [60, Theorem 10.4(f)], respectively. Nevertheless, our methods can detect non-formality, even in this delicate setting.

**Example 8.2.** Consider the arrangement $\mathcal{A} = \mathcal{A}(31425)$ defined in complex coordinates by the function $Q(z, w) = z(z - w)(z - 2w)(2z + 3w - 5\overline{w})(2z - w - 5\overline{w})$; see [51, Example 6.5]. A computation shows that $TC_1(\mathcal{V}_2(G))$ has 9 irreducible components, while $\mathcal{R}_2(G)$ has 10 irreducible components; see [52, Example 10.2], and [51, Example 6.5], respectively. By Theorem 4.13, the group $G$ is not 1-formal. Thus, the complement $M$ cannot be a formal space, despite a claim to the contrary by Ziegler [75, p. 10].

## 9. Configuration spaces

Denote by $S^{\times n}$ the $n$-fold cartesian product of a connected space $S$. Consider the *configuration space* of $n$ distinct labeled points in $S$,

$$F(S, n) = S^{\times n} \setminus \bigcup_{i<j} \Delta_{ij},$$

where $\Delta_{ij}$ is the diagonal $\{s \in S^{\times n} \mid s_i = s_j\}$. The topology of configuration spaces has attracted considerable attention over the years. For $S$ a smooth, complex projective variety, the cohomology algebra $H^*(F(S, n), \mathbb{C})$ has been described by Totaro [70], solely in terms of $n$ and the cohomology algebra $H^*(S, \mathbb{C})$.

Let $C_g$ be a smooth compact complex curve of genus $g$ ($g \geq 1$). The fundamental group of the configuration space $M_{g,n} := F(C_g, n)$ may be identified with $P_{g,n}$, the pure braid group on $n$ strings of the underlying Riemann surface. Starting from Totaro's description, it is straightforward to check that the low-degrees cup-product map of $P_{g,n}$ is equivalent, in the sense of Definition 5.8, to the composite

$$(9.1) \quad \mu_{g,n} \colon \bigwedge^2 H^1(C_g^{\times n}, \mathbb{C}) \xrightarrow{\cup_{C_g^{\times n}}} H^2(C_g^{\times n}, \mathbb{C}) \twoheadrightarrow H^2(C_g^{\times n}, \mathbb{C}) / \operatorname{span}\{[\Delta_{ij}]\}_{i<j} \,,$$

where $[\Delta_{ij}] \in H^2(C_g^{\times n}, \mathbb{C})$ denotes the dual class of the diagonal $\Delta_{ij}$, and the second arrow is the canonical projection. It follows that the connected smooth quasi-projective complex variety $M_{g,n}$ has the property that $W_1(H^1(M_{g,n}, \mathbb{C})) = H^1(M_{g,n}, \mathbb{C})$, for all $g, n \geq 1$.

The Malcev Lie algebra of $P_{g,n}$ has been computed by Bezrukavnikov in [6], for all $g, n \geq 1$. It turns out that the groups $P_{g,n}$ are 1-formal, for $g > 1$ and $n \geq 1$, or $g = 1$ and $n \leq 2$; see [6, p. 130]. On the other hand, Bezrukavnikov also states in [6, Proposition 4.1(a)] that $P_{1,n}$ is not 1-formal for $n \geq 3$, without giving a full argument. With our methods, this can be easily proved.

**Example 9.1.** Let $\{a, b\}$ be the standard basis of $H^1(C_1, \mathbb{C}) = \mathbb{C}^2$. Note that the cohomology algebra $H^*(C_1^{\times n}, \mathbb{C})$ is isomorphic to $\bigwedge^*(a_1, b_1, \dots, a_n, b_n)$. Denote by $(x_1, y_1, \dots, x_n, y_n)$ the coordinates of $z \in H^1(P_{1,n}, \mathbb{C})$. Using (9.1), it is readily seen that

$$\mathcal{R}_1(P_{1,n}) = \left\{ (x, y) \in \mathbb{C}^n \times \mathbb{C}^n \;\middle|\; \begin{array}{l} \sum_{i=1}^n x_i = \sum_{i=1}^n y_i = 0, \\ x_i y_j - x_j y_i = 0, \text{ for } 1 \leq i < j < n \end{array} \right\}.$$

Suppose $n \geq 3$. Then $\mathcal{R}_1(P_{1,n})$ is a rational normal scroll in $\mathbb{C}^{2(n-1)}$, see [34], [26]. In particular, $\mathcal{R}_1(P_{1,n})$ is an irreducible, non-linear variety. From Theorem 6.5, we conclude that $P_{1,n}$ is indeed non-1-formal. This indicates that Theorem 1.3 from [33] cannot hold in the stated generality.

This family of examples also shows that both the $\mathcal{R}_1$–version of Arapura's result on $\mathcal{V}_1$ from Theorem 5.2(1) and the resonance obstruction test from our Theorem 6.5(4) may fail, for an arbitrary smooth quasi-projective variety $M$.

For $n \leq 2$, things are even simpler.

**Example 9.2.** It follows from (9.1) that $\mu_{1,2}$ equals the canonical projection

$$\mu_{1,2} \colon \bigwedge^2(a_1, b_1, a_2, b_2) \twoheadrightarrow \bigwedge^2(a_1, b_1, a_2, b_2)/\mathbb{C} \cdot (a_1 - a_2)(b_1 - b_2).$$

It follows that $\mathcal{R}_1(P_{1,2})$ is a 2-dimensional, 0-isotropic linear subspace of $H^1(P_{1,2}, \mathbb{C})$.

Consider now the smooth variety $M_g' := M_{1,2} \times C_g$, with $g \geq 2$. By Proposition 7.2, this variety has 1-formal fundamental group. It also has the property that $W_1(H^1(M_g', \mathbb{C})) = H^1(M_g', \mathbb{C})$. We infer from Lemma 7.5 that

$$\mathcal{R}_1(\pi_1(M_g')) = \mathcal{R}_1(P_{1,2}) \times \{0\} \cup \{0\} \times H^1(C_g, \mathbb{C}),$$

where the component $\mathcal{R}_1(P_{1,2})$ is 0-isotropic and the component $H^1(C_g, \mathbb{C})$ is 1-isotropic. We thus see that both cases listed in Proposition 5.10(1) may actually occur.

**Remark 9.3.** Recall from Example 5.11 that the tangent cone formula may fail for quasi-projective groups, at least in the case when 1 is an isolated point of the characteristic variety. The following statement can be extracted from [44, p. 161]:

"If $M$ is a quasi-projective variety and 1 is not an isolated point of $\mathcal{V}_1(\pi_1(M))$, then $TC_1(\mathcal{V}_1(\pi_1(M))) = \mathcal{R}_1(\pi_1(M))$." Taking $M$ to be one of the configuration spaces $M_{1,n}$, with $n \geq 3$, shows that this statement does not hold, even when $M$ is smooth.

Indeed, since $P_{1,2}$ is 1-formal, we obtain from Theorem A that $\mathcal{V}_1(P_{1,2})$ is 2-dimensional at 1. As is well-known, the natural surjection, $P_{1,n} \twoheadrightarrow P_{1,2}$, embeds $\mathcal{V}_1(P_{1,2})$ into $\mathcal{V}_1(P_{1,n})$, for $n \geq 2$. Thus, $\mathcal{V}_1(P_{1,n})$ is positive-dimensional at 1, for $n \geq 2$. On the other hand, it follows from Example 9.1 that $TC_1(\mathcal{V}_1(P_{1,n}))$ is strictly contained in $\mathcal{R}_1(P_{1,n})$, for $n \geq 3$.

## 10. Artin groups

In this section, we analyze the class of finite-type Artin groups. Using the resonance obstructions from Theorem B, we give a complete answer to Serre's question for right-angled Artin groups, and we give a Malcev Lie algebra version of the answer for arbitrary Artin groups.

10.1. **Labeled graphs and Artin groups.** Let $\Gamma = (\mathsf{V}, \mathsf{E}, \ell)$ be a labeled finite simplicial graph, with vertex set $\mathsf{V}$, edge set $\mathsf{E} \subset \binom{\mathsf{V}}{2}$, and labeling function $\ell\colon \mathsf{E} \to \mathbb{N}_{\geq 2}$. Finite simplicial graphs are identified in the sequel with labeled finite simplicial graphs with $\ell(e) = 2$, for each $e \in \mathsf{E}$.

**Definition 10.2.** The *Artin group* associated to the labeled graph $\Gamma$ is the group $G_\Gamma$ generated by the vertices $v \in \mathsf{V}$ and with a defining relation

$$\underbrace{vwv\cdots}_{\ell(e)} = \underbrace{wvw\cdots}_{\ell(e)}$$

for each edge $e = \{v, w\}$ in $\mathsf{E}$. If $\Gamma$ is unlabeled, then $G_\Gamma$ is called a *right-angled Artin group*, and is defined by commutation relations $vw = wv$, one for each edge $\{v, w\} \in \mathsf{E}$.

**Example 10.3.** Let $\Gamma = (\mathsf{V}, \mathsf{E}, \ell)$ and $\Gamma' = (\mathsf{V}', \mathsf{E}', \ell')$ be two labeled graphs. Denote by $\Gamma \bigsqcup \Gamma'$ their disjoint union, and by $\Gamma * \Gamma'$ their *join*, with vertex set $\mathsf{V} \bigsqcup \mathsf{V}'$, edge set $\mathsf{E} \bigsqcup \mathsf{E}' \bigsqcup \{\{v, v'\} \mid v \in \mathsf{V}, v' \in \mathsf{V}'\}$, and label 2 on each edge $\{v, v'\}$. Then

$$G_{\Gamma \bigsqcup \Gamma'} = G_\Gamma * G_{\Gamma'} \quad \text{and} \quad G_{\Gamma * \Gamma'} = G_\Gamma \times G_{\Gamma'}\,.$$

In particular, if $\Gamma$ is a discrete graph, i.e., $\mathsf{E} = \emptyset$, then $G_\Gamma = \mathbb{F}_n$, whereas if $\Gamma$ is an (unlabeled) complete graph, i.e., $\mathsf{E} = \binom{\mathsf{V}}{2}$, then $G_\Gamma = \mathbb{Z}^n$, where $n = |\mathsf{V}|$. More generally, if $\Gamma$ is a complete multipartite graph (i.e., a finite join of discrete graphs), then $G_\Gamma$ is a finite direct product of finitely generated free groups.

Given a graph $\Gamma = (\mathsf{V}, \mathsf{E})$ and a subset of vertices $\mathsf{W} \subset \mathsf{V}$, we denote by $\Gamma(\mathsf{W})$ the full subgraph of $\Gamma$, with vertex set $\mathsf{W}$ and edge set $\mathsf{E} \cap \binom{\mathsf{W}}{2}$.

Let $(S^1)^{\mathsf{V}}$ be the compact $n$-torus, where $n = |\mathsf{V}|$, endowed with the standard cell structure. Denote by $K_\Gamma$ the subcomplex of $(S^1)^{\mathsf{V}}$ having a $k$-cell for each subset $\mathsf{W} \subset \mathsf{V}$ of size $k$ for which $\Gamma(\mathsf{W})$ is a complete graph. As shown by Charney–Davis

[11] and Meier–VanWyk [53], $K_\Gamma = K(G_\Gamma, 1)$. In particular, the cup-product map $\cup_{G_\Gamma}\colon H^1(G_\Gamma,\mathbb{C}) \wedge H^1(G_\Gamma,\mathbb{C}) \to H^2(G_\Gamma,\mathbb{C})$ may be identified with the linear map $\cup_\Gamma\colon \mathbb{C}^{\mathsf{V}} \wedge \mathbb{C}^{\mathsf{V}} \to \mathbb{C}^{\mathsf{E}}$ defined by

$$v \cup_\Gamma w = \begin{cases} \pm\{v,w\}, & \text{if } \{v,w\} \in \mathsf{E},\\ 0, & \text{otherwise,}\end{cases} \tag{10.1}$$

with signs determined by fixing an orientation of the edges of $\Gamma$.

10.4. **Jumping loci for right-angled Artin groups.** The resonance varieties of right-angled Artin groups were described explicitly in Theorem 5.5 from [61]. If $\Gamma = (\mathsf{V}, \mathsf{E})$ is a graph, then

$$\mathcal{R}_1(G_\Gamma) = \bigcup_{\mathsf{W}} \mathbb{C}^{\mathsf{W}}, \tag{10.2}$$

where the union is taken over all subsets $\mathsf{W} \subset \mathsf{V}$ such that $\Gamma(\mathsf{W})$ is disconnected, and maximal with respect to this property. Moreover, the decomposition (10.2) coincides with the decomposition into irreducible components of $\mathcal{R}_1(G_\Gamma)$.

Before proceeding to the Serre problem, we describe the characteristic variety of $G_\Gamma$. For $\mathsf{W} \subset \mathsf{V}$, define the subtorus $\mathbb{T}_{\mathsf{W}} \subset \mathbb{T}_{G_\Gamma} = (\mathbb{C}^*)^{\mathsf{V}}$ by

$$\mathbb{T}_{\mathsf{W}} = \{(t_v)_{v\in \mathsf{V}} \in (\mathbb{C}^*)^{\mathsf{V}} \mid t_v = 1 \text{ for } v \notin \mathsf{W}\}.$$

The map $\exp\colon T_1\mathbb{T}_{G_\Gamma} \to \mathbb{T}_{G_\Gamma}$ is the componentwise exponential map $\exp^{\mathsf{V}}\colon \mathbb{C}^{\mathsf{V}} \to (\mathbb{C}^*)^{\mathsf{V}}$; its restriction to the subspace spanned by $\mathsf{W}$ is $\exp^{\mathsf{W}}\colon \mathbb{C}^{\mathsf{W}} \to (\mathbb{C}^*)^{\mathsf{W}} = \mathbb{T}_{\mathsf{W}}$.

**Proposition 10.5.** *Let $G_\Gamma$ be the right-angled Artin group associated to the graph $\Gamma = (\mathsf{V}, \mathsf{E})$. Then*

$$\mathcal{V}_1(G_\Gamma) = \bigcup_{\mathsf{W}} \mathbb{T}_{\mathsf{W}},$$

*where the union is over all subsets $\mathsf{W} \subset \mathsf{V}$ such that $\Gamma(\mathsf{W})$ is maximally disconnected. Moreover, this decomposition coincides with the decomposition into irreducible components of $\mathcal{V}_1(G_\Gamma)$.*

*Proof.* The realization of $K(G_\Gamma, 1)$ as a subcomplex $K_\Gamma$ of the torus $(S^1)^{\mathsf{V}}$ yields a well-known resolution of the trivial $\mathbb{Z}G_\Gamma$-module $\mathbb{Z}$ by finitely generated, free $\mathbb{Z}G_\Gamma$-modules, as the augmented, $G_\Gamma$-equivariant chain complex of the universal cover of $K_\Gamma$,

$$\widetilde{C}_\bullet(\widetilde{K_\Gamma})\colon \ \cdots \longrightarrow \mathbb{Z}G_\Gamma \otimes C_k \xrightarrow{d_k} \mathbb{Z}G_\Gamma \otimes C_{k-1} \longrightarrow \cdots \xrightarrow{d_1} \mathbb{Z}G_\Gamma \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0\ .$$

Here $C_k$ denotes the free abelian group generated by the $k$-cells of $K_\Gamma$, and the boundary maps are given by

$$d_k(e_{v_1} \times \cdots \times e_{v_k}) = \sum_{i=1}^{k} (-1)^{i-1}(v_i - 1) \otimes e_{v_1} \times \cdots \times \widehat{e}_{v_i} \times \cdots \times e_{v_k}, \tag{10.3}$$

where, for each $v \in \mathsf{V}$, the symbol $e_v$ denotes the 1-cell corresponding to the $v$-th factor of $(S^1)^{\mathsf{V}}$.

Now let $\rho = (t_v)_{v\in\mathsf{V}} \in (\mathbb{C}^*)^{\mathsf{V}}$ be an arbitrary character. Denoting by $\{v^*\}_{v\in\mathsf{V}}$ the basis of $H^1(G_\Gamma, \mathbb{C})$ dual to the canonical basis of $H_1(G_\Gamma, \mathbb{C})$, define an element $z \in \mathbb{C}^{\mathsf{V}} = H^1(G_\Gamma, \mathbb{C})$ by $z = \sum_{v\in\mathsf{V}}(t_v - 1)v^*$. Using (10.3), it is not difficult to check the following equality of cochain complexes

$$\mathrm{Hom}_{\mathbb{Z}G_\Gamma}(\widetilde{C}_\bullet(\widetilde{K_\Gamma}), {}_\rho\mathbb{C}) = (H^\bullet(G_\Gamma, \mathbb{C}), \mu_z). \tag{10.4}$$

It follows then, directly from the definitions (4.6) and (4.2), and using (10.4), that $\rho \in \mathcal{V}_1(G_\Gamma)$ if and only if $z \in \mathcal{R}_1(G_\Gamma)$. Hence, the claimed decomposition of $\mathcal{V}_1(G_\Gamma)$ is a direct consequence of the decomposition (10.2). □

10.6. **Serre's problem for right-angled Artin groups.** As shown by Kapovich and Millson in [38, Theorem 16.10], all Artin groups are 1-formal. This opens the way for approaching Serre's problem for Artin groups via resonance varieties, which, as noted above, were described explicitly in [61]. Using these tools, we find out precisely which right-angled Artin groups can be realized as fundamental groups of quasi-compact Kähler manifolds.

**Theorem 10.7.** *Let $\Gamma = (\mathsf{V}, \mathsf{E})$ be a finite simplicial graph, with associated right-angled Artin group $G_\Gamma$. The following are equivalent.*

(i) *The group $G_\Gamma$ is quasi-Kähler.*
(ii) *The isotropicity property from Definition 6.1 is satisfied by $\cup_{G_\Gamma}$.*
(iii) *The graph $\Gamma$ is complete multipartite graph.*
(iv) *The group $G_\Gamma$ is a product of finitely generated free groups.*

*Proof.* For the implication (i) ⇒ (ii), use the 1-formality of $G_\Gamma$ and Theorem B.

The implication (ii) ⇒ (iii) is proved by induction on $n = |\mathsf{V}|$. If $\Gamma$ is complete, then $\Gamma$ is the join of $n$ graphs with one vertex. Otherwise, there is a subset $\mathsf{W} \subset \mathsf{V}$ such that $\Gamma(\mathsf{W})$ is disconnected, and maximal with respect to this property. Write $\mathsf{W} = \mathsf{W}' \bigsqcup \mathsf{W}''$, with both $\mathsf{W}'$ and $\mathsf{W}''$ non-empty and with no edge connecting $\mathsf{W}'$ to $\mathsf{W}''$. Then $\Gamma(\mathsf{W}) = \Gamma(\mathsf{W}') \bigsqcup \Gamma(\mathsf{W}'')$, and so $G_{\Gamma(\mathsf{W})} = G_{\Gamma(\mathsf{W}')} * G_{\Gamma(\mathsf{W}'')}$. Hence, $w' \cup_{\Gamma(\mathsf{W})} w'' = 0$, for any $w' \in \mathsf{W}'$ and $w'' \in \mathsf{W}''$. We infer from [61, Lemma 5.2] that $\mathcal{R}_1(G_{\Gamma(\mathsf{W})}) = \mathbb{C}^{\mathsf{W}}$.

On the other hand, we know from (10.2) that $\mathbb{C}^{\mathsf{W}}$ is a positive-dimensional irreducible component of $\mathcal{R}_1(G_\Gamma)$. Our hypothesis implies that $\mathbb{C}^{\mathsf{W}}$ is either 0-isotropic or 1-isotropic with respect to $\cup_{\Gamma(\mathsf{W})}$. By Lemma 7.4, $\cup_{\Gamma(\mathsf{W}')} = \cup_{\Gamma(\mathsf{W}'')} = 0$. The cup-product formula (10.1) implies that $\Gamma(\mathsf{W})$ is a discrete graph.

If $\mathsf{W} = \mathsf{V}$, we are done. Otherwise, $\mathsf{V} = \mathsf{W} \bigsqcup \mathsf{W}_1$, with $|\mathsf{W}_1| < n$. Since $\Gamma(\mathsf{W})$ is maximally disconnected, this forces $\{w, w_1\} \in \mathsf{E}$, for all $w \in \mathsf{W}$ and $w_1 \in \mathsf{W}_1$. In other words, $\Gamma$ is the join $\Gamma(\mathsf{W}) * \Gamma(\mathsf{W}_1)$; thus, $G_\Gamma = G_{\Gamma(\mathsf{W})} \times G_{\Gamma(\mathsf{W}_1)}$. By Lemma 7.5, $\cup_{\Gamma(\mathsf{W}_1)}$ inherits from $\cup_\Gamma$ the isotropicity property. This completes the induction.

The implication (iii) ⇒ (iv) follows from the discussion in Example 10.3.

Finally, the implication (iv) ⇒ (i) follows by taking products and realizing free groups by the complex line with a number of points deleted. □

As is well-known, two right-angled Artin groups are isomorphic if and only if the corresponding graphs are isomorphic. Evidently, there are infinitely many graphs which are not joins of discrete graphs. Thus, implication (i) ⇒ (iii) from Theorem 10.7 allows us to recover, in sharper form, a result of Kapovich and Millson (Theorem 14.7 from [38]).

**Corollary 10.8.** *Among right-angled Artin groups $G_\Gamma$, there are infinitely many mutually non-isomorphic groups which are not isomorphic to fundamental groups of smooth, quasi-projective complex varieties.*

10.9. **A Malcev Lie algebra version of Serre's question.** Next, we describe a construction that associates to a labeled graph $\Gamma = (\mathsf{V}, \mathsf{E}, \ell)$ an ordinary graph, $\tilde{\Gamma} = (\tilde{\mathsf{V}}, \tilde{\mathsf{E}})$, which we call the *odd contraction* of $\Gamma$. First define $\Gamma_{\mathrm{odd}}$ to be the unlabeled graph with vertex set $\mathsf{V}$ and edge set $\{e \in \mathsf{E} \mid \ell(e) \text{ is odd}\}$. Then define $\tilde{\mathsf{V}}$ to be the set of connected components of $\Gamma_{\mathrm{odd}}$, with two distinct components determining an edge $\{c, c'\} \in \tilde{\mathsf{E}}$ if and only if there exist vertices $v \in c$ and $v' \in c'$ which are connected by an edge in $\mathsf{E}$.

**Example 10.10.** Let $\Gamma$ be the complete graph on vertices $\{1, 2, \dots, n-1\}$, with labels $\ell(\{i,j\}) = 2$ if $|i-j| > 1$ and $\ell(\{i,j\}) = 3$ if $|i-j| = 1$. The corresponding Artin group is the classical braid group on $n$ strings, $B_n$. Since in this case $\Gamma_{\mathrm{odd}}$ is connected, the odd contraction $\tilde{\Gamma}$ is the discrete graph with a single vertex.

**Lemma 10.11.** *Let $\Gamma = (\mathsf{V}, \mathsf{E}, \ell)$ be a labeled graph, with odd contraction $\tilde{\Gamma} = (\tilde{\mathsf{V}}, \tilde{\mathsf{E}})$. Then the Malcev Lie algebra of $G_\Gamma$ is filtered Lie isomorphic to the Malcev Lie algebra of $G_{\tilde{\Gamma}}$.*

*Proof.* The Malcev Lie algebra of $G_\Gamma$ was computed in [38, Theorem 16.10]. It is the quotient of the free Malcev Lie algebra on $\mathsf{V}$, $\widehat{\mathbb{L}}(\mathsf{V})$, by the closed Lie ideal generated by the differences $u - v$, for odd-labeled edges $\{u, v\} \in \mathsf{E}$, and by the brackets $[u, v]$, for even-labeled edges $\{u, v\} \in \mathsf{E}$. Plainly, this quotient is filtered Lie isomorphic to the quotient of $\widehat{\mathbb{L}}(\tilde{\mathsf{V}})$ by the closed Lie ideal generated by the brackets $[c, c']$, for $\{c, c'\} \in \tilde{\mathsf{E}}$, which is just the Malcev Lie algebra of $G_{\tilde{\Gamma}}$. □

The Coxeter group associated to a labeled graph $\Gamma = (\mathsf{V}, \mathsf{E}, \ell)$ is the quotient of the Artin group $G_\Gamma$ by the normal subgroup generated by $\{v^2 \mid v \in \mathsf{V}\}$. If the Coxeter group $W_\Gamma$ is finite, then $G_\Gamma$ is quasi-projective. The proof of this assertion, due to Brieskorn [7], involves the following steps: $W_\Gamma$ acts as a group of reflections in some $\mathbb{C}^n$; the action is free on the complement $M_\Gamma$ of the arrangement of reflecting hyperplanes of $W_\Gamma$, and $G_\Gamma = \pi_1(M_\Gamma / W_\Gamma)$; finally, the quotient manifold $M_\Gamma / W_\Gamma$ is a complex smooth quasi-projective variety.

It would be interesting to know exactly which (non-right-angled) Artin groups can be realized by smooth, quasi-projective complex varieties. We give an answer to this question, albeit only at the level of Malcev Lie algebras of the respective groups.

**Corollary 10.12.** *Let $\Gamma$ be a labeled graph, with associated Artin group $G_\Gamma$ and odd contraction the unlabeled graph $\tilde{\Gamma}$. The following are equivalent.*

(i) *The Malcev Lie algebra of $G_\Gamma$ is filtered Lie isomorphic to the Malcev Lie algebra of a quasi-Kähler group.*
(ii) *The isotropicity property from Definition 6.1 is satisfied by $\cup_{G_\Gamma}$.*
(iii) *The graph $\tilde{\Gamma}$ is a complete multipartite graph.*
(iv) *The Malcev Lie algebra of $G_\Gamma$ is filtered Lie isomorphic to the Malcev Lie algebra of a product of finitely generated free groups.*

*Proof.* By Lemma 10.11, the Malcev Lie algebras of $G_\Gamma$ and $G_{\tilde{\Gamma}}$ are filtered isomorphic. Hence, the graded Lie algebras $\operatorname{gr}^*(G_\Gamma)\otimes\mathbb{C}$ and $\operatorname{gr}^*(G_{\tilde{\Gamma}})\otimes\mathbb{C}$ are isomorphic.

From [68], we know that the kernel of the Lie bracket, $\bigwedge^2 \operatorname{gr}^1(G)\otimes\mathbb{C}\to\operatorname{gr}^2(G)\otimes\mathbb{C}$, is equal to $\operatorname{im}(\partial_G)$, for any finitely presented group $G$. It follows that the cup-product maps $\cup_{G_\Gamma}$ and $\cup_{G_{\tilde{\Gamma}}}$ are equivalent, in the sense of Definition 5.8. Consequently, $\cup_{G_\Gamma}$ satisfies the second resonance obstruction if and only if $\cup_{G_{\tilde{\Gamma}}}$ does so.

With these remarks, the Corollary follows at once from Theorems 10.7 and 6.5. □

10.13. **Kähler right-angled Artin groups.** With our methods, we may easily decide which right-angled Artin groups are Kähler groups.

**Corollary 10.14.** *For a right-angled Artin group $G_\Gamma$, the following are equivalent.*

(i) *The group $G_\Gamma$ is Kähler.*
(ii) *The graph $\Gamma$ is a complete graph on an even number of vertices.*
(iii) *The group $G_\Gamma$ is a free abelian group of even rank.*

*Proof.* Implications (ii) ⇒ (iii) ⇒ (i) are clear. So suppose $G_\Gamma$ is a Kähler group. By Theorem 10.7, $\Gamma$ is a complete multi-partite graph $\overline{K}_{n_1}*\cdots*\overline{K}_{n_r}$, and $G_\Gamma = \mathbb{F}_{n_1}\times\cdots\times\mathbb{F}_{n_r}$. By Lemma 7.5, and abusing notation slightly, $\mathcal{R}_1(G_\Gamma)=\bigcup_i \mathcal{R}_1(\mathbb{F}_{n_i})$. Now, if there were an index $i$ for which $n_i>1$, then $\mathcal{R}_1(\mathbb{F}_{n_i})=\mathbb{C}^{n_i}$ would be a positive-dimensional, 0-isotropic, irreducible component of $\mathcal{R}_1(G_\Gamma)$, contradicting Corollary 6.7(1). Thus, we must have $n_1=\cdots=n_r=1$, and $\Gamma=K_r$. Moreover, since $G_\Gamma=\mathbb{Z}^r$ is a Kähler group, $r$ must be even. □

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